

On the Moduli Space of the Cascading $SU(M + p) \times SU(p)$ Gauge Theory

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We carry out a thorough analysis of the moduli space of the cascading gauge theory found on p D3-branes and M wrapped D5-branes at the tip of the conifold. We find various mesonic branches of the moduli space whose string duals involve the warped deformed conifold with different numbers of mobile D3-branes. The branes that are not mobile form a BPS bound state at threshold. In the special case where p is divisible by M there also exists a one-dimensional baryonic branch whose family of supergravity duals, the resolved warped deformed conifolds, was constructed recently. The warped deformed conifold is a special case of these backgrounds where the resolution parameter vanishes and a \mathbb{Z}_2 symmetry is restored. We study various brane probes on the resolved warped deformed conifolds, and successfully match the results with the gauge theory. In particular, we show that the radial potential for a D3-brane on this space varies slowly, suggesting a new model of D-brane inflation.

1. Introduction

Consideration of p D3-branes at the tip of the conifold leads to the duality conjecture [1] relating type IIB string theory on $AdS_5 \times T^{11}$ to a superconformal $SU(p) \times SU(p)$ gauge theory. Addition of M D5-branes wrapped over the two-cycle of T^{11} deforms the gauge group to $SU(M+p) \times SU(p)$ [2] and breaks the conformal invariance, producing a logarithmic running of the gauge couplings [3]. This theory exhibits a “duality cascade” [4,5] where along the RG flow p repeatedly drops by M units as a result of the duality of [6].

The complete and non-singular supergravity dual of the cascading gauge theory, the warped deformed conifold, was found in [5]. In the infrared it exhibits confinement and chiral symmetry breaking, while in the UV there is a logarithmic running of coupling constants and a duality cascade. In the absence of extra branes, this background is dual to the $SU(M(k+1)) \times SU(Mk)$ theory at a special \mathbb{Z}_2 -symmetric point on the baryonic branch $\mathcal{AB} = \text{const}$ where the two baryonic condensates are equal, $|\mathcal{A}| = |\mathcal{B}|$ [5,7,8]. The cascading gauge theory has a pseudoscalar Goldstone mode of the spontaneously broken $U(1)_{\text{baryon}}$, and its massless scalar superpartner [7]. The supergravity duals of these modes were found in [8]. The scalar zero-mode, which produces a small motion along the baryonic branch was found with the help of the Papadopoulos-Tseytlin ansatz [9] generalizing the $SO(4)$ and \mathbb{Z}_2 symmetric warped deformed conifold to include a breaking of the \mathbb{Z}_2 . A general analysis of the supersymmetry conditions for this ansatz led to a derivation [10] of coupled first-order equations describing the entire baryonic branch of confining vacua. This family of *resolved warped deformed conifolds* is then readily constructed through numerical integration of the equations of [10] subject to the requirement that at large radius they asymptote to the cascading solution of [4].

On the $SU(N_1) \times SU(N_2)$ gauge theory side, an analysis of various branches of the moduli space was begun in [5], and continued in [11]. In this paper we carry out a complete analysis and compare it successfully with the dual string theory. The branches are labelled by two integers. One of them, $r = 1, \dots, M$, is associated with a spontaneous breaking of the \mathbb{Z}_{2M} R-symmetry to \mathbb{Z}_2 . The other, $l = 0, \dots, k = [p/M]$, has the following interpretation: $M(l+1)$ D5-branes and lM anti-D5-branes which wrap the two-cycle of $T^{1,1}$ form a bound state at threshold. The remaining $p - lM$ D3-branes are free to move on the deformed conifold whose deformation parameter ϵ depends on r and l . In the special case where p is a multiple of M there exists a branch with no mobile D3-branes, $l = k$. This branch

breaks the baryon number symmetry of the problem and will be referred to as a baryonic branch. It is reminiscent of the baryonic branch of [12]. This picture of the moduli space follows from a careful field theory analysis using standard techniques (for a review, see e.g. [13]), but a few interesting subtleties which have so far been ignored turn out to be quite important.

For the specific example of the baryonic branch, which exists only for $N_1 = M(k + 1)$, $N_2 = Mk$, we carefully impose the boundary conditions on the numerical solution, and show that they lead to a constant tension of the BPS domain wall of [14-16] along the entire branch. We also calculate the tensions of various other objects, the confining string, the D3 and anti-D3 branes, and find that they blow up far along the branch. We find that small departures from the \mathbb{Z}_2 symmetric point create a small potential for a D3-brane which depends on the radial coordinate of the classical solution. This suggests a string theoretic mechanism for brane inflation where a D3-brane rolls towards smaller radius on a resolved warped deformed conifold embedded into a flux compactification. This approach is similar to that of KKLMNT [17], but instead of an anti-D3 brane uses a Fayet-Iliopoulos parameter ξ [18] to resolve the warped deformed conifold and generate a potential for the D3-brane. Our proposal is therefore similar to the D-term inflation mechanism of [19,20]. D-terms also play an important role in string theoretic constructions involving D7-branes and D3-branes [21].

The paper is organized as follows. In section 2 we review the gauge theory and its symmetries. In section 3 we state our main result for the quantum structure of the moduli space and discuss its D-brane interpretation. Section 4 is devoted to the classical analysis of the mesonic and the baryonic branches of the moduli space. In section 5 we analyze the $SU(p + M) \times SU(p)$ gauge theory with $p < M$, and find the quantum deformation of the classical mesonic branches. In section 6 we study $p = M$ at the quantum level, and find a mesonic and a baryonic branch. In sections 7 and 8 we study the quantum moduli spaces for $p = M + 1$ and $p > M + 1$, respectively. In section 9 we discuss how the different theories and different branches are related by Higgsing and duality transformations. This leads to non-trivial consistency checks of our quantitative results. In section 10 we summarize our results on the gauging of $U(1)_{baryon}$ and turning on the Fayet-Iliopoulos parameter ξ .

In section 11 we compare various gauge theory and corresponding string theory objects. We match the BPS and non-BPS domain walls present in the gauge theory with D5-branes and NS5-branes wrapped over the three-sphere at the bottom of the deformed conifold. We present a general argument showing that the tension of BPS domain walls

is independent of the moduli. We also comment on how the tensions of confining strings, glueballs, and solitonic strings depend on the continuous parameter $g_s M$ present in the cascading gauge theory. In sections 12 and 13 we review and present some new results on the resolved warped deformed conifolds, which are supergravity duals of the baryonic branch. We solve the equations derived in [10], while carefully imposing the boundary conditions at large radius. In section 14 we check the consistency of our numerical solutions by showing that the BPS domain wall tension is constant along the branch; we also study the tensions of confining strings and of anti-D3-branes along the baryonic branch. In section 15 we study the potential generated for D3-branes and suggest a string theoretic implementation of the D-term inflation. Some possible extensions of our work are mentioned in the Discussion. Appendices A and B contain some further details about the resolved warped deformed conifold.

2. The gauge theory

In this section we consider the gauge dynamics of the supersymmetric field theory with gauge group

$$SU(N_1 = M + p) \times SU(N_2 = p) \quad (2.1)$$

with $p \geq 0$ (clearly, $N_1 \geq N_2$). We parameterize it as

$$N_1 = (k + 1)M + \tilde{p} \quad ; \quad N_2 = kM + \tilde{p} \quad ; \quad \tilde{p} = 0, \dots, M - 1 \quad (2.2)$$

We add matter fields

$$\begin{aligned} A_{\alpha i}^a & \text{ in } (\mathbf{N}_1, \overline{\mathbf{N}}_2) , \\ B_{\dot{\alpha} a}^i & \text{ in } (\overline{\mathbf{N}}_1, \mathbf{N}_2) \end{aligned} \quad (2.3)$$

$(\alpha, \dot{\alpha} = 1, 2, i = 1, \dots, N_1, a = 1, \dots, N_2)$, and a tree level superpotential

$$W_0 = h \text{Tr}_a \det_{\alpha \dot{\alpha}} A_{\alpha} B_{\dot{\alpha}} = h (A_{1i}^a B_{1b}^i A_{2j}^b B_{2a}^i - A_{1i}^a B_{2b}^i A_{2j}^b B_{1a}^i) . \quad (2.4)$$

This gauge theory describes N_1 D5-branes and N_2 anti-D5-branes wrapping the collapsed \mathbb{S}^2 at the singularity of the conifold \mathcal{C}_0 . \mathcal{C}_0 is parameterized by four complex numbers $z_{\alpha \dot{\alpha}}$ subject to the equation $\det_{\alpha \dot{\alpha}} z_{\alpha \dot{\alpha}} = 0$.

The $SU(N_1)$ ($SU(N_2)$) gauge theory has $2N_2$ ($2N_1$) flavors. Therefore, if the superpotential (2.4) is ignored, the instanton factors of these two gauge groups are

$$\Lambda_1^{3N_1 - 2N_2} = \Lambda_1^{3M + p} \quad ; \quad \Lambda_2^{3N_2 - 2N_1} = \Lambda_2^{p - 2M} \quad (2.5)$$

Let us discuss the global symmetries of this theory. Clearly, there is an $SU(2) \times SU(2)$ symmetry which acts on the indices α and $\dot{\alpha}$. The global Abelian symmetry can be analyzed in the basis

$$\begin{array}{cccc}
& U(1)_A & U(1)_B & U(1)_R \\
A & 1 & 0 & 1 \\
B & 0 & 1 & 1 \\
h & -2 & -2 & -2 \\
\Lambda_1^{3M+p} & 2p & 2p & 2(M+p) \\
\Lambda_2^{p-2M} & 2(M+p) & 2(M+p) & 2p
\end{array} \tag{2.6}$$

The exact symmetry of the system is the subgroup of (2.6) which is not broken by nonzero h and the anomalies. It is $U(1)_{baryon} \times \mathbb{Z}_{2M}$. $U(1)_{baryon}$ is generated by the difference of the $U(1)_A$ and $U(1)_B$ generators, and \mathbb{Z}_{2M} is an R-symmetry which is generated by $A \rightarrow e^{\frac{2\pi i}{2M}} A$, $B \rightarrow e^{\frac{2\pi i}{2M}} B$, and $\theta \rightarrow e^{\frac{2\pi i}{2M}} \theta$. Below we will also discuss the effect of gauging $U(1)_{baryon}$ and adding a Fayet-Iliopoulos parameter ξ . On the string theory side this happens when the gauge theory is embedded into a flux compactification with a compact Calabi-Yau space.

We will find it convenient to form the following combinations of the parameters

$$\begin{array}{ccc}
& U(1)_A & U(1)_B & U(1)_R \\
L_1(M, p) = h^p \Lambda_1^{3M+p} & 0 & 0 & 2M \\
L_2(M, p) = h^{M+p} \Lambda_2^{p-2M} & 0 & 0 & -2M \\
I(M, p) = L_1(M, p) L_2(M, p) & 0 & 0 & 0
\end{array} \tag{2.7}$$

which do not transform under $U(1)_A \times U(1)_B$.

The combination $I(M, p)$ in (2.7) which is invariant under all the symmetries is dimensionless, and therefore it is invariant under the renormalization group. It has a natural interpretation in the brane system as the instanton factor of the type IIB string theory

$$I(M, p) = L_1(M, p) L_2(M, p) = h^{M+2p} \Lambda_1^{3M+p} \Lambda_2^{p-2M} = e^{2\pi i \tau} \tag{2.8}$$

One aspect of this interpretation is that $I(M, p) = e^{2\pi i \tau}$ is the amplitude of a type IIB D-instanton. It is related to two fractional D-instantons on the conifold corresponding the instantons of the two gauge groups, $SU(N_1)$ and $SU(N_2)$. This explains why $I(M, p)$ includes the instanton factors of the two gauge groups.

The ratio $L_1(M, p)/L_2(M, p)$ is determined by the NS-NS and RR two form potentials through the two sphere:

$$\frac{L_1(M, p)}{L_2(M, p)} \sim \exp \left[\frac{1}{\pi \alpha'} \int_{\mathbb{S}^2} (B_2/g_s + iC_2) \right]. \tag{2.9}$$

In the conformal case $M = 0$, similar relations between gauge theory and string theory parameters were proposed in [1].

3. Summary of the gauge theory results

To facilitate the reading of the paper, we summarize here our conclusions about the moduli space of vacua of the quantum field theory. For more details, see sections 4 - 11.

Our conclusion will be that the classical and quantum moduli spaces of vacua are quite different. The quantum moduli space is

$$\oplus_{r=1}^M \oplus_{l=0}^k \text{Sym}_{p-lM}(\mathcal{C}_{r,l}) \quad (3.1)$$

Here $\mathcal{C}_{r,l}$ is the deformed conifold which is smooth. It is described by an equation in four complex variables $z_{\alpha\dot{\alpha}}$

$$\det_{\alpha\dot{\alpha}} z_{\alpha\dot{\alpha}} = \epsilon \quad (3.2)$$

The $\mathcal{C}_{r,l}$ arising on different branches of (3.1) have different deformation parameters $\epsilon_{M,P}(r,l)$ (see below). In the special case of $\tilde{p} = 0$, the last term $\text{Sym}_{p-kM=\tilde{p}=0}(\mathcal{C}_{r,l=k})$ is replaced by \mathbb{C} .

The sum over r in (3.1) reflects the spontaneous breaking of the global \mathbb{Z}_{2M} R-symmetry to \mathbb{Z}_2 , which is achieved through gluino condensation in a low energy $SU(M)$ subgroup. This $SU(M)$ group arises differently on different branches in (3.1) and in different regions of the moduli space on the same branch. Sometimes it is a nontrivial subgroup of the microscopic $SU(N_1) \times SU(N_2)$, while in other cases it involves dual gauge groups. Its instanton factor is

$$\Lambda(M,p,l)^{3M} \sim h^{p+l(M+2p)} \Lambda_1^{(3M+p)(l+1)} \Lambda_2^{(p-2M)l} = L_1(M,p) I(M,p)^l \quad (3.3)$$

and its gluino condensation leads to the low energy superpotential

$$W(M,p,l,r) = M \left(\Lambda(M,p,l)^{3M} \right)^{\frac{1}{M}} = M \left(L_1(M,p)^{3M} \right)^{\frac{1}{M}} \left(I(M,p) \right)^{\frac{l}{M}} \quad (3.4)$$

Above we used the invariant combination $I(M,p)$ of (2.8), and we suppressed the phase $e^{\frac{2\pi i r}{M}}$ which arises from the branch of the fractional power and leads to the r dependence. Note that the l dependence is only through the power of I . Below we will see in detail how the superpotential (3.4) is generated; it arises differently on different branches.

In our case the parameters $z_{\alpha\dot{\alpha}}$ arise from eigenvalues of the matrix

$$\sqrt{h} \mathcal{M}_{\alpha\dot{\alpha}b}^a = \sqrt{h} A_{\alpha i}^a B_{\dot{\alpha}b}^i \quad (3.5)$$

where the factor of \sqrt{h} is introduced to simplify the equations. We will see that the deformation parameter ϵ depends on the branch in (3.1),

$$\epsilon_{M,p}(r,l) \sim (\Lambda(M,p,l)^{3M})^{\frac{1}{M}} \sim \epsilon_{M,p}(r,l=0)I(M,p)^{\frac{1}{M}} \quad (3.6)$$

The label r in $\epsilon_{M,p}(r,l)$ is the branch of the fractional power $e^{\frac{2\pi r i}{M}}$ (to simplify the equation we suppress this factor).

In terms of the D-brane interpretation, the space $Sym_{p-lM}(\mathcal{C}_{r,l})$ describes $p-lM$ pairs of D5-anti-D5-branes each forming a D3-brane which leaves the tip of the conifold and is free to move in its bulk after it is deformed to $\mathcal{C}_{r,l}$. The remaining $(l+1)M$ D5-branes and lM anti-D5-branes form a *bound state at threshold*. The deformation by (3.6) with its M branches labelled by r in (3.1) is generated by the strong coupling $SU(M)$ dynamics with scale $\Lambda(M,p,l)$ of the D5-branes near the tip.

The branch of the moduli space in (3.1) with $Sym_{p-lM}(\mathcal{C}_{r,l})$ will turn out to have the massless photons of $U(1)^{p-lM-1}$ – one fewer than the number of mobile D3-branes. If $U(1)_{baryon}$ is gauged, then there is one more $U(1)$ factor and the low energy spectrum is that of $p-lM$ multiplets of $\mathcal{N} = 4$. This is consistent because the global $U(1)_{baryon}$ symmetry is not broken on all these branches.

An important exception to this is the last term $Sym_0(\mathcal{C}_{r,l})$ which appears only when $\tilde{p} = 0$. In this case $U(1)_{baryon}$ is broken and hence $Sym_0(\mathcal{C}_{r,l})$ should be replaced by a copy of \mathbb{C} . This is the baryonic branch. If $U(1)_{baryon}$ is gauged, then it is Higgsed, and this branch is lifted [8,18]. In this case \mathbb{C} is replaced by a point, and the pattern is more uniform. In section 10 we will discuss the effect of turning on a Fayet-Iliopoulos parameter ξ .

4. Classical flat directions

In this section we examine the classical moduli space of vacua. The D-term equations set

$$\begin{aligned} \sum_{\alpha} A_{\alpha} A_{\alpha}^{\dagger} - \sum_{\dot{\alpha}} B_{\dot{\alpha}}^{\dagger} B_{\dot{\alpha}} &= \frac{\mathcal{U}}{p} \mathbb{I}_p \\ \sum_{\alpha} A_{\alpha}^{\dagger} A_{\alpha} - \sum_{\dot{\alpha}} B_{\dot{\alpha}} B_{\dot{\alpha}}^{\dagger} &= \frac{\mathcal{U}}{M+p} \mathbb{I}_{M+p} \end{aligned} \quad (4.1)$$

where \mathbb{I}_p and \mathbb{I}_{M+p} are $p \times p$ and $(M+p) \times (M+p)$ unit matrices, and

$$\mathcal{U} = \text{Tr} \left(\sum_{\alpha} A_{\alpha} A_{\alpha}^{\dagger} - \sum_{\dot{\alpha}} B_{\dot{\alpha}}^{\dagger} B_{\dot{\alpha}} \right) . \quad (4.2)$$

In the quantum theory, \mathcal{U} is an operator, whose expectation value labels different ground states. Gauging $U(1)_{baryon}$ sets $\mathcal{U} = 0$ in (4.1). However, in this case a Fayet-Iliopoulos term ξ can be added, and then (4.1) has

$$\mathcal{U} = \xi . \quad (4.3)$$

The solutions of these equations together with the F-term equations depend on the values of p and M . We find two kinds of classical solutions which we refer to as mesonic and baryonic.

4.1. Mesonic flat direction

We always have the mesonic flat directions (up to gauge transformations)

$$\begin{aligned} A_\alpha &= \begin{pmatrix} A_{\alpha 1}^1 & & & & \\ & A_{\alpha 2}^2 & & & \\ & & A_{\alpha 3}^3 & & \\ & & & \ddots & \\ & & & & A_{\alpha p}^p \end{pmatrix} \\ B_{\dot{\alpha}}^{tr} &= \begin{pmatrix} B_{\dot{\alpha} 1}^1 & & & & \\ & B_{\dot{\alpha} 2}^2 & & & \\ & & B_{\dot{\alpha} 3}^3 & & \\ & & & \ddots & \\ & & & & B_{\dot{\alpha} p}^p \end{pmatrix} \\ \sum_{\alpha} |A_{\alpha a}^a|^2 - \sum_{\dot{\alpha}} |B_{\dot{\alpha} a}^a|^2 &= 0 \quad \forall a \end{aligned} \quad (4.4)$$

At a generic point along this moduli space of vacua the gauge group is broken to $SU(M) \times U(1)^{p-1}$. The moduli space can be characterized by p sets of coordinates $z_{\alpha\dot{\alpha}}^a = A_{\alpha a}^a B_{\dot{\alpha} a}^a$ with $\det_{\alpha\dot{\alpha}} z_{\alpha\dot{\alpha}}^a = 0$ up to permutations over the index a . This is a symmetric product of p copies of the (singular) conifold \mathcal{C}_0

$$Sym_p(\mathcal{C}_0) \quad (4.5)$$

In addition to the $SU(M)$ gauge multiplets, the low energy spectrum at a generic point includes $3p$ chiral multiplets and $p - 1$ vector multiplets.

If $U(1)_{baryon}$ is gauged, we have in addition to the $SU(M)$ part, p multiplets of $\mathcal{N} = 4$. If we also add a Fayet-Iliopoulos term ξ for the $U(1)_{baryon}$, then supersymmetry is broken

in all these vacua. To leading order in the $U(1)_{baryon}$ gauge coupling g , the vacua are given by (4.4) and the vacuum energy is simply

$$V = \frac{g^2 \xi^2}{2} . \quad (4.6)$$

Even though this section is devoted mainly to the classical physics of the model, we would like to make some comments about the semi-classical low energy dynamics of the unbroken $SU(M)$ gauge theory. It is a subgroup of the original $SU(N_1 = M + p)$ and its instanton factor is

$$\Lambda_1^{3M+p} h^p \sim \Lambda(M, P, l = 0)^{3M} \quad (4.7)$$

This expression agrees with our general expression for the scale of the unbroken group (3.3). Equation (4.7) follows from matching factors from the Higgsing $SU(M+p) \rightarrow SU(M)$ and from the fields which get masses proportional to h . It is consistent with the (anomalous) symmetries of the problem (2.6).

The nonperturbative dynamics of the low energy gauge group leads to gluino condensation and to a superpotential

$$W = M \Lambda(M, P, l = 0)^{\frac{1}{M}} = M (\Lambda_1^{3M+p} h^p)^{\frac{1}{M}} \quad (4.8)$$

as in (3.4). The M branches of (4.8) lead to M vacua for each point in (4.5). This is the origin of the sum over r in the quantum moduli space (3.1). Note that since (4.7) and (4.8) are independent of the moduli in (4.4), the flat directions are not lifted. Instead, as we will see below, the moduli space (4.5) is deformed. Even though the superpotential (4.8) does not lead to a potential, it is important for determining the tensions of domain walls connecting the different M vacua [14-16] (see section 11).

Let us examine the limit where $A_{\alpha p}^p$ and $B_{\alpha p}^p$ are much bigger than all other entries in (4.4). Then, the low energy theory is an $SU(N_1 - 1 = M + p - 1) \times SU(N_2 - 1 = p - 1) \times U(1)$ gauge theory with matter fields A and B , as in (2.3), and three neutral chiral multiplets, whose vevs are $A_{\alpha p}^p B_{\alpha p}^p$. This is the same as the original theory except that p is reduced to $p - 1$, we have three more neutral chiral fields and a $U(1)$ factor. The instanton factors of $SU(N_1 - 1 = M + p - 1) \times SU(N_2 - 1 = p - 1)$ are related to the original ones (2.5) as

$$\hat{\Lambda}_1^{3M+p-1} \sim \Lambda_1^{3M+p} h \quad ; \quad \hat{\Lambda}_2^{p-2M-1} \sim \Lambda_2^{p-2M} h \quad (4.9)$$

Note that these relations are independent of the vev of $A_{\alpha p}^p B_{\alpha p}^p$. Iterating this equation p times leads to (4.7). The $U(1)$ factor originates from the two gauge groups $U(1) \subset SU(N_1) \times SU(N_2)$. It is important that the massless components of A and B are charged under this $U(1)$. Therefore, in this low energy $SU(N_1 - 1) \times SU(N_2 - 1) \times U(1)$ theory, the $U(1)_{baryon}$ is gauged.

4.2. *Baryonic flat directions for $\tilde{p} = 0$, i.e. $N_1 = (k+1)M$, $N_2 = kM$*

For $\tilde{p} = 0$ we find, in addition to the mesonic branch (4.5), two baryonic flat directions:

$$\begin{aligned}
A_{\alpha=1} &= C \begin{pmatrix} \sqrt{k} & 0 & 0 & \cdot & 0 & 0 \\ 0 & \sqrt{k-1} & 0 & \cdot & 0 & 0 \\ 0 & 0 & \sqrt{k-2} & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & 1 & 0 \end{pmatrix} \\
A_{\alpha=2} &= C \begin{pmatrix} 0 & 1 & 0 & \cdot & 0 & 0 \\ 0 & 0 & \sqrt{2} & \cdot & 0 & 0 \\ 0 & 0 & 0 & \sqrt{3} & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & \sqrt{k} \end{pmatrix}
\end{aligned} \tag{4.10}$$

$$B_{\dot{\alpha}=1} = 0$$

$$B_{\dot{\alpha}=2} = 0$$

and another branch with $A \longleftrightarrow B$. Here C is an arbitrary complex number and each entry in the matrices is an $M \times M$ unit matrix. The real constant \mathcal{U} in (4.1) is given by $\mathcal{U} = k(k+1)M|C|^2$ for the solution (4.10) and by $\mathcal{U} = -k(k+1)M|C|^2$ for the solution with $A \longleftrightarrow B$. The gauge invariant operator which is nonzero along these branches is the baryon $(A_1 A_2)^{k(k+1)M/2}$ (with appropriate contraction of the indices) or the anti-baryon $(B_1 B_2)^{k(k+1)M/2}$. Each of these branches is one complex dimensional and is labelled by C (more precisely, by $C^{k(k+1)M}$), and they touch each other at the origin, $C = 0$. We refer to these branches as baryonic because, in contrast to the mesonic branch (4.4), here $U(1)_{baryon}$ is broken for nonzero C . In fact, as we will see, these two branches are joined in the quantum theory into a single smooth branch.

The low energy theory along each baryonic branch includes a chiral superfield C and an unbroken $SU(M)$. Unlike the unbroken $SU(M)$ on the mesonic branch, here $SU(M) \subset SU(M)_{N_1} \times SU(M)_{N_2}$ where $SU(M)_{N_1} \subset SU(M)^{k+1} \subset SU(N_1 = (k+1)M)$ and $SU(M)_{N_2} \subset SU(M)^k \subset SU(N_2 = kM)$; i.e. the index of the embedding in $SU(N_1)$ is $k+1$ and the index of the embedding in $SU(N_2)$ is k . The $SU(M)$ instanton factor is not given by (4.7), but by

$$\Lambda_1^{(k+1)(k+3)M} \Lambda_2^{k(k-2)M} h^{2k(k+1)M} \sim \Lambda(M, p = kM, l = k)^{3M} \tag{4.11}$$

Here we used the index of the embedding in the two groups and the contribution from the matter fields which acquired mass from the superpotential W_0 . Again, note that this

relation is independent of the modulus C . This agrees with our general expression for the scale of the unbroken group (3.3). Again, gluino condensation leads to a superpotential $M\Lambda(M, p = kM, l = k)^{\frac{1}{M}}$ and hence to M vacua. Since (4.11) is independent of the modulus C , the flat direction is not lifted.

If $U(1)_{baryon}$ is gauged, C must vanish, and it seems that the theory is at the origin of field space. We will see below that this is not true in the quantum theory where in this case the theory has isolated vacua with broken $U(1)_{baryon}$. One way to see that is to add a Fayet-Iliopoulos term ξ for $U(1)_{baryon}$. Then, depending on the sign of ξ either A or B is nonzero as in (4.10) and C is fixed in terms of ξ . More explicitly, for positive ξ we have the solution (4.10) with

$$\xi = \mathcal{U} = k(k+1)M|C|^2. \quad (4.12)$$

Since $C \neq 0$, the gauged $U(1)_{baryon}$ symmetry is Higgsed, the phase of C acquires a mass, and the vacuum is isolated.

Now we return to the case where $U(1)_{baryon}$ is not gauged. For $\tilde{p} \neq 0$ the only flat directions are the mesonic ones, (4.4). One way to understand it is by trying to reduce p using expectation values as in (4.4) down to $p = kM$, and looking for a baryonic solution similar to (4.10) using the massless fields. More explicitly, consider (4.4) with $A_{\alpha a}^a = B_{\alpha a}^a = 0$ for $a = 1, \dots, kM$. The low energy theory is an $SU((k+1)M) \times SU(kM) \times U(1)^{\tilde{p}}$ gauge theory with charged fields as in (2.3), and neutral chiral fields taking values in \tilde{p} copies of \mathcal{C}_0 . The charged matter in this theory is very similar to the that of the $\tilde{p} = 0$ theory which leads to (4.10), except for one important difference. The charged chiral fields are charged under one linear combination of $U(1)^{\tilde{p}}$. As we commented above, in this case C in (4.10) must vanish, so this does not lead to new flat directions. Below we will see how this fact is modified in the quantum theory.

5. $SU(N_1 = M + p) \times SU(N_2 = p)$ **with** $p = 0, \dots, M - 1$

Here we discuss the quantum theory for small values of p . As we have already commented, in the quantum theory the $SU(M)$ that remains unbroken along the flat directions becomes strong and leads to M vacua. Hence, the previous answer for the moduli space appears M times. Yet, this is not the whole story.

In order to examine it in more detail we first ignore the tree level superpotential, W_0 , and the weaker of the two gauge groups, $SU(p)$. Since the first group, $SU(N_1 = M + p)$,

has fewer flavors than colors ($2N_2 = 2p < N_1$), its flat directions are characterized by the meson fields

$$\mathcal{M}_{\alpha\dot{\alpha}a}^b = A_{\alpha i}^a B_{\dot{\alpha}i}^b. \quad (5.1)$$

At a generic point along these flat directions, the unbroken gauge symmetry is $SU(N_1 - 2N_2 = M - p) \subset SU(N_1 = M + p)$. This gauge theory confines and its dynamics generates the superpotential [22,23]

$$W_{dyn} = (N_1 - 2N_2) \left(\frac{\Lambda_1^{3N_1-2N_2}}{\det_{\alpha\dot{\alpha}ab} \mathcal{M}} \right)^{\frac{1}{N_1-2N_2}} = (M - p) \left(\frac{\Lambda_1^{3M+p}}{\det_{\alpha\dot{\alpha}ab} \mathcal{M}} \right)^{\frac{1}{M-p}} \quad (5.2)$$

The fractional power in the superpotential is associated with the $M - p$ vacua of $SU(M - p)$. Therefore, we should study the dynamical superpotential as a function on an $M - p$ fold cover of the space of \mathcal{M} .

So far our description has neglected the tree level superpotential W_0 and the D-term equations of the second gauge group, $SU(N_2 = p)$. Therefore, \mathcal{M} is generic. On the other hand, the typical points on the classical flat directions (4.4) have non-generic \mathcal{M} , such that $\det_{\alpha\dot{\alpha}ab} \mathcal{M} = 0$. For generic \mathcal{M} the $SU(N_1)$ gauge symmetry is broken to $SU(N_1 - 2N_2 = M - p)$, while along the classical flat directions of the full theory (4.4) the $SU(N_1)$ gauge group is broken to $SU(M)$. We will now show that restoring W_0 and the $SU(N_2 = p)$ interactions restricts us to a subspace of \mathcal{M} . This subspace is not the same as the classical moduli space (4.4); it is a deformation of it.

Consider a generic point in \mathcal{M} . After the unbroken $SU(N_1 - 2N_2 = M - p)$ confines and leads to (5.2), the low energy theory is an $SU(N_2 = p)$ gauge theory with neutral matter fields \mathcal{M} of (5.1) (four adjoints and four singlets), and the superpotential

$$W_{eff} = W_0 + W_{dyn} \quad (5.3)$$

This theory is IR free and can be analyzed easily.

Solving $\partial_{\mathcal{M}} W_{eff} = 0$ we find

$$\begin{aligned} h^p \det_{\alpha\dot{\alpha}ab} \mathcal{M} &\sim \left(\Lambda_1^{3M+p} h^p \right)^{\frac{p}{M}} \\ h \text{Tr}_a \det_{\alpha\dot{\alpha}} \mathcal{M}_{\alpha\dot{\alpha}} &\sim \left(h^p \Lambda_1^{3M+p} \right)^{\frac{1}{M}} \end{aligned} \quad (5.4)$$

Although classically $\det \mathcal{M} = 0$, in the quantum theory this is deformed by nonperturbative effects. As a check, note that in the $\Lambda_1 \rightarrow 0$ limit the determinant vanishes. Related to

that is the appearance of a fractional power $\frac{1}{M}$. It reflects the fact that $SU(N_1)$ is broken to $SU(M)$ with no charged matter with scale

$$\Lambda_1^{3M+p} h^p \sim \Lambda(M, P, l=0)^{3M}, \quad (5.5)$$

which agrees with our general expression for the unbroken group and its instanton factor (3.3) and (4.7). The strong dynamics of this $SU(M)$ theory leads to M vacua.

Finally, in addition to (5.4) we also need to impose the D-term equations of $SU(N_2)$. The conclusion is that the moduli space can be parameterized by the $4p$ numbers $\mathcal{M}_{\alpha\dot{\alpha}a}^a$ with $a = 1, \dots, p$, $\alpha, \dot{\alpha} = 1, 2$ subject to the constraints

$$h \det_{\alpha\dot{\alpha}} \mathcal{M}_{\alpha\dot{\alpha}a}^a = \epsilon_{M,p}(r, l=0) \sim \left(h^p \Lambda_1^{3M+p} \right)^{\frac{1}{M}} \quad \forall a \quad (5.6)$$

i.e. they are on the deformed conifold $\mathcal{C}_{r,l=0}$ (3.2) with ϵ as in (3.6). Of course, we should mod out this space by the permutation of the p points, and therefore the moduli space is $\oplus_r \text{Sym}_p(\mathcal{C}_{r,l=0})$.

Classically, the answer (4.5) is related to the conifold \mathcal{C}_0 . The nonperturbative effects associated with a power of Λ_1 (5.6) deform it to the deformed conifold $\mathcal{C}_{r,l}$. More precisely, we work on an $M - p$ fold cover of the space of \mathcal{M} and find M different solutions of (5.4).

The final answer is

$$\oplus_{r=1}^M \text{Sym}_p(\mathcal{C}_{r,l=0}) \quad (5.7)$$

At a generic point in this space the $SU(p)$ is broken to $U(1)^{p-1}$. The $3p$ massless chiral multiplets describe the positions of p D3-branes in the deformed conifold $\mathcal{C}_{r,l}$.¹

The answer (5.7) can be interpreted as p pairs of D5-anti-D5-branes moving from the tip to the bulk of the deformed conifold $\mathcal{C}_{r,l}$ as p D3-branes. The different phases of ϵ are different fluxes through the cycle of the deformed conifold. To reach this conclusion we needed the dynamical superpotential in (5.3) – the tree level theory based on the superpotential W_0 does not lead to this answer.

To summarize, the answer (5.7) differs from the classical answer derived from (4.4) in two ways. First, we have M branches originating from the $SU(M)$ gauge dynamics. Second, the argument of the symmetric product \mathcal{C}_0 is deformed to $\mathcal{C}_{r,l=0}$. As expected, far from the origin of the moduli space the classical analysis, together with the existence of M vacua in the strongly coupled low energy theory, gives a good approximation to the quantum answer.

¹ If $U(1)_{baryon}$ is gauged, the low energy modes combine into p vector multiplets of $\mathcal{N} = 4$.

6. $SU(N_1 = 2M) \times SU(N_2 = M)$; i.e. $p = M$

In this case the $SU(N_1)$ theory has equal numbers of flavors and colors. Therefore, in addition to the mesons $\mathcal{M}_{\alpha\dot{\alpha}a}^b$ of (5.1) which are four adjoints and four singlets of $SU(N_2)$, the theory has baryons

$$\begin{aligned}\mathcal{A} &= \epsilon^{i_1 \dots i_{N_1}} A_{\alpha_1 i_1}^{a_1} \dots A_{\alpha_{N_1} i_{N_1}}^{a_{N_1}} , \\ \mathcal{B} &= \epsilon_{i_1 \dots i_{N_1}} B_{\dot{\alpha}_1 a_1}^{i_1} \dots B_{\dot{\alpha}_{N_1} a_{N_1}}^{i_{N_1}} .\end{aligned}\tag{6.1}$$

Bose statistics shows that these are singlets of the non-Abelian symmetries in the problem [24], and in particular of $SU(N_2)$ and $SU(2) \times SU(2)$ [5,7,8]. The fields \mathcal{M} , \mathcal{A} and \mathcal{B} are not independent. They are subject to the constraint [24] $\det_{\alpha\dot{\alpha}ab} \mathcal{M} - \mathcal{A}\mathcal{B} = \Lambda_1^{2N_1}$ where we have already taken the nonperturbative quantum corrections into account. This can be summarized by an effective superpotential [24]

$$W_{eff} = W_0 + L \left(\det_{\alpha\dot{\alpha}ab} \mathcal{M} - \mathcal{A}\mathcal{B} - \Lambda_1^{2N_1} \right)\tag{6.2}$$

where L is a Lagrange multiplier.² The low energy theory is based on an $SU(N_2)$ gauge theory with these massless fields and the superpotential (6.2). It is IR free and is easily analyzed.

The moduli space has two branches which are related to the branches of the classical moduli space.

The mesonic branch has $l = 0$. It is characterized by $\mathcal{A} = \mathcal{B} = 0$ and \mathcal{M} constrained by $\det_{\alpha\dot{\alpha}ab} \mathcal{M} = \Lambda_1^{2N_1}$. It is also subject to the $SU(N_2)$ D-term equations and stationarity of W_0 . This leads to the moduli space $\oplus_r \text{Sym}_{p=M}(\mathcal{C}_{r,l=0})$ with the deformation parameter $\epsilon_{M,p=M}(r, l=0)$ of (3.6), as in (5.6). At a generic point on this branch the theory has $M - 1$ vector multiplets and M chiral multiplets corresponding to the motion of the M D3-branes on $\mathcal{C}_{r,l=0}$.³

The second branch is baryonic. It has $\mathcal{M} = 0$ and $\mathcal{A}\mathcal{B} = \Lambda_1^{2N_1}$. As discussed above, the low energy theory includes a pure gauge $SU(M)$ sector which leads to M vacua. Each

² It is important to clarify a common misconception. The superpotential (6.2) is not a low energy superpotential. It includes fields which are massive everywhere on the moduli space, in particular, the Lagrange multiplier L and one component of \mathcal{M} , \mathcal{A} and \mathcal{B} are always massive. These fields are not associated with massive particles in the spectrum. Instead, they should be interpreted as auxiliary fields in the low energy theory.

³ If we gauge the $U(1)_{baryon}$, we find $p = M$ multiplets of $\mathcal{N} = 4$.

of the M components of the baryonic branch is one complex dimensional. Note that the two classical branches with $\mathcal{AB} = 0$ are combined into a single smooth branch with $\mathcal{AB} = \Lambda_1^{2N_1}$.

If we gauge $U(1)_{baryon}$, the baryonic branch becomes zero dimensional. It has M discrete points – a point for each component of the baryonic branch labelled by $r = 1, \dots, M$. Each of them has a gap because the $U(1)_{baryon}$ gauge symmetry is Higgsed [8]. The existence of such points is completely quantum mechanical. They originate from the singularity at the origin of the classical moduli space. In the quantum theory this singularity leads to several vacua. The mesonic branch which classically touches the origin is deformed. In addition, the origin leads to these isolated vacua where $U(1)_{baryon}$ is broken. We now turn on a nonzero Fayet-Iliopoulos term ξ . As above, supersymmetry is broken in the mesonic branch. The M isolated vacua remain supersymmetric, but their position in field space changes. As $\xi \rightarrow \infty$ these supersymmetric vacua move to large field strength where they are continuously connected to the classical vacua discussed around (4.12).

In different regions of the baryonic branch the low energy $SU(M)$ gauge theory and its scale can be understood differently. First, far out along the baryonic branch (4.10) we can use the general result (4.11). Specializing to $k = 1$, it is

$$\Lambda_1^{8M} \Lambda_2^{-M} h^{4M} \sim \Lambda(M, p = M, l = 1)^{3M} \quad (6.3)$$

Near the origin, where $SU(N_1 = 2M)$ is strongly coupled, this calculation is not valid. Instead, we can find $\Lambda(M, p = M, l = 1)$ as follows. Above the scale Λ_1 the second gauge group $SU(N_2 = M)$ has $4M$ fundamental flavors, and its instanton factor is Λ_2^{-M} . Below Λ_1 the fundamental flavors are confined and they are replaced by four adjoints \mathcal{M} . The instanton factor Λ_2^{-M} does not change. Then the adjoints get mass $h\Lambda_1^2$ and the resulting instanton factor is $\Lambda_2^{-M} (h\Lambda_1^2)^{4M}$. Miraculously, this different calculation which is based on different physics agrees with (6.3), and also with the general expression (3.3).

We have already stated that the expectation values of the mesons are interpreted as D3-branes in the bulk of the conifold. We would like to interpret the baryonic branch as describing *a BPS bound state at threshold of $2M$ D5-branes and M anti-D5-branes*. If $U(1)_{baryon}$ is not gauged, the baryon number is broken by the bound state and it is part of a one complex dimensional branch of the moduli space.

7. $SU(N_1 = 2M + 1) \times SU(N_2 = M + 1)$; i.e. $p = M + 1$

We now consider the case $p = M + 1$. Here the $SU(N_1)$ gauge theory interacts with $2N_2 = 2M + 2 = N_1 + 1$ flavors; i.e. it is the number of colors plus one. Its dynamics leads to the mesons \mathcal{M} of (5.1) and baryons $\mathcal{A}_{\alpha a}$, $\mathcal{B}_{\dot{\alpha}}^a$ similar to (6.1) in that they have opposite charges under the baryon number symmetry $U(1)_{baryon}$. However, now they are in the anti-fundamental and the fundamental of $SU(N_2)$ respectively [24]. Each of them transforms under one of the two factors in $SU(2) \times SU(2)$.

The low energy theory includes these mesons, baryons and the gauge fields of $SU(N_2)$ with a superpotential [24]

$$W_{eff} = W_0 - \frac{1}{\Lambda_1^{3N_1 - 2N_2}} \left(\det_{\alpha\dot{\alpha}ab} \mathcal{M} - \mathcal{A}\mathcal{M}\mathcal{B} \right) \quad (7.1)$$

The moduli space is easily determined and again has two branches. On one of them, $\mathcal{A} = \mathcal{B} = 0$ and \mathcal{M} satisfies (5.4) as well as the $SU(p)$ D-term equations. This leads to $\oplus_r Sym_{p=M+1}(\mathcal{C}_{r,l=0})$ with the deformation parameter ϵ as in (3.6) and (5.6)

$$\epsilon_{M,p=M+1}(r, l=0) \sim (h^{M+1} \Lambda_1^{4M+1})^{\frac{1}{M}} \quad (7.2)$$

On the other branch \mathcal{M} is massive and should be integrated out. Clearly

$$\mathcal{M}_{\alpha\dot{\alpha}a}^b \sim \frac{1}{h\Lambda_1^{4M+1}} \epsilon_{\alpha\dot{\alpha}} \epsilon_{\beta\dot{\beta}} \mathcal{B}_{\dot{\beta}}^b \mathcal{A}_{\beta a} \quad (7.3)$$

($\mathcal{O}((\mathcal{B}\mathcal{A})^2)$ corrections from $\det \mathcal{M}$ in (7.1) are not present because \mathcal{M} of (7.3) has low rank.) This leads to a superpotential of the form

$$\frac{1}{h\Lambda_1^{8M+2}} \det_{\alpha\dot{\alpha}} (\mathcal{A}_{\alpha a} \mathcal{B}_{\dot{\alpha}}^a) \quad (7.4)$$

In addition, the low energy theory has an $SU(p = M + 1)$ gauge theory under which \mathcal{A} and \mathcal{B} transform.

This theory is similar to the one discussed in the section 5 about $p < M$, if we use

$$\hat{M} = M \quad ; \quad \hat{p} = 1 \quad (7.5)$$

i.e. it has gauge group $SU(\hat{N}_1 = \hat{M} + \hat{p} = M + 1)$ and $\hat{N}_2 = 1$. Therefore, we can borrow the result of the analysis there. In order to do that we have to bring it to a canonical form and relate its parameters to the parameters of the “hat theory.”

First, the “quarks” \mathcal{A}_α and $\mathcal{B}_{\dot{\alpha}}$ do not have their canonical dimensions. Therefore we define the canonical fields $\hat{\mathcal{A}}_\alpha = \mathcal{A}_\alpha/\Lambda_1^{2M}$ and $\hat{\mathcal{B}}_{\dot{\alpha}} = \mathcal{B}_{\dot{\alpha}}/\Lambda_1^{2M}$. Their quartic coupling is

$$\hat{h} = \frac{1}{h\Lambda_1^2} \quad (7.6)$$

The scale of the $SU(N_2 = M + 1)$ is also modified. The instanton factor of the high energy theory is $\Lambda_2^{p-2M} = \Lambda_2^{1-M}$. Below the scale Λ_1 the $SU(N_2)$ gauge theory has two fundamental flavors \mathcal{A} and \mathcal{B} and four adjoints \mathcal{M} . Therefore, its instanton factor is $\Lambda_L^{3M+3-2-4(M+1)} = \Lambda_L^{-3-M} \sim \Lambda_2^{1-M}\Lambda_1^{-4}$. Then, at lower energies the four adjoints get a mass of order $h\Lambda_1^2$ and the instanton factor of the $SU(M + 1)$ gauge theory is

$$\hat{\Lambda}_1^{3M+1} \sim \Lambda_L^{-3-M} (h\Lambda_1^2)^{4(M+1)} \sim \Lambda_2^{1-M} h^{4(M+1)} \Lambda_1^{8M+4} \quad (7.7)$$

Now it is straightforward to use the results from section 5. The moduli space is M copies of $\mathcal{C}_{r,l}$ with

$$\hat{h} \det_{\alpha\dot{\alpha}} \hat{\mathcal{A}}_{\alpha\alpha} \hat{\mathcal{B}}_{\dot{\alpha}}^a \sim \hat{\epsilon}_{\hat{M}=M, \hat{p}=1}(r, l=0) \sim (\hat{h} \hat{\Lambda}_1^{3M+1})^{\frac{1}{M}} \quad (7.8)$$

Expressing it in terms of \mathcal{A} and \mathcal{B} and finally in terms of the eigenvalues of $\sqrt{h}\mathcal{M}$ we find the deformed conifold with deformation parameter

$$\epsilon_{M,p=M+1}(r, l=1) \sim (h^{4M+3} \Lambda_1^{8M+2} \Lambda_2^{1-M})^{\frac{1}{M}} \quad (7.9)$$

Note the combination

$$h^{4M+3} \Lambda_1^{8M+2} \Lambda_2^{1-M} \sim \Lambda(M, p=M+1, l=1)^{3M} \quad (7.10)$$

which can be interpreted as the instanton factor of the low energy $SU(M)$ theory. The expressions (7.9)(7.10) are in agreement with our general expression (3.3)(3.6). Finally, the low energy spectrum has three chiral superfields.⁴

We would like to make a few comments about this branch. First, even though \mathcal{A} and \mathcal{B} carry baryon number, their expectation values do not mean that $U(1)_{baryon}$ is broken. Instead, $U(1)_{baryon}$ combined with a broken gauge symmetry is unbroken. One way to see that is to note that \mathcal{A} and \mathcal{B} are not gauge invariant. The order parameter along the moduli space is $\mathcal{A}_{\alpha\alpha} \mathcal{B}_{\dot{\alpha}}^a$ which is $U(1)_{baryon}$ neutral. Equivalently, the relation (7.3) along

⁴ If we gauge $U(1)_{baryon}$, the low energy spectrum forms a $\mathcal{N} = 4$ supermultiplet.

the moduli space shows that baryon number is not broken, and therefore we cannot refer to this branch as a baryonic branch.

Second, we should clarify the relation to the classical analysis in section 4. Equation (7.3) shows that in the classical limit $\Lambda_1 \rightarrow 0$ the order parameters \mathcal{A} and \mathcal{B} must vanish in order to have finite \mathcal{M} . This shows that the semiclassical limit of these vacua is given by \mathcal{M} with rank one. We have discussed these vacua when we looked for classical baryonic branches of this theory. We separated one eigenvalue of the meson field in order to have a low energy theory with $p = M$ which has a baryonic branch. There we saw that the low energy theory has a gauged baryon number, and therefore there is no classical baryonic branch. As we stated above, even when $U(1)_{baryon}$ is gauged, the quantum $p = M$ theory has isolated vacua in which $U(1)_{baryon}$ is Higgsed. The branch we have been discussing here in the $p = M + 1$ theory is associated with such a mesonic eigenvalue and such an isolated vacuum of the low energy theory. In terms of the branes on the conifold, it is a single D3-brane in the bulk of $\mathcal{C}_{r,l=1}$ and a bound state at threshold of the other branes.

Finally, we would like to point out a subtlety in integrating out \mathcal{M} . There is no problem with doing it along the $l = 1$ branch, as in (7.3), (7.4). (Note that this integration out obscures the semi-classical limit.) However, such integration out is impossible along the mesonic branch with $l = 0$, because there the field \mathcal{M} has massless components. We should stress though that (7.1) is valid on both branches.

In conclusion, our moduli space is

$$\oplus_{r=1}^M [Sym_{M+1}(\mathcal{C}_{r,l=0}) \oplus \mathcal{C}_{r,l=1}] \quad (7.11)$$

The first term is interpreted as $p = M + 1$ D3-branes on the deformed conifold. On the second branch there is a BPS bound state at threshold of $2M$ D5-branes and M anti-D5-branes while the remaining D3-brane is free to move on the deformed conifold $\mathcal{C}_{r,l=1}$.

8. $SU(N_1 = M + p) \times SU(N_2 = p)$ **with** $M + 1 < p$

The previous case of $p = M + 1$ has almost all the elements the we need for the general case of larger p .

The first branch of the moduli space is a deformation of (4.4). As above, we first ignore the $SU(N_2)$ dynamics and the tree level superpotential ($h = 0$). The $SU(N_1)$ dynamics has many more flavors than colors, and therefore the rank of the meson \mathcal{M} is constrained.

Nevertheless, some of the dynamics can be recovered by considering \mathcal{M} of maximal rank $2p$ and the superpotential (5.2)

$$W_{dyn} = (N_1 - 2N_2) \left(\frac{\Lambda_1^{3N_1-2N_2}}{\det_{\alpha\dot{\alpha}ab} \mathcal{M}} \right)^{\frac{1}{N_1-2N_2}} = (M - p) \left(\frac{\Lambda_1^{3M+p}}{\det_{\alpha\dot{\alpha}ab} \mathcal{M}} \right)^{\frac{1}{M-p}} \quad (8.1)$$

As in all our cases, solving $\partial_{\mathcal{M}} W_{eff} = 0$ we find (5.4)

$$\begin{aligned} h^p \det_{\alpha\dot{\alpha}ab} \mathcal{M} &\sim \left(\Lambda_1^{3M+p} h^p \right)^{\frac{p}{M}} \\ h \text{Tr}_a \det_{\alpha\dot{\alpha}} \mathcal{M}_{\alpha\dot{\alpha}} &\sim \left(h^p \Lambda_1^{3M+p} \right)^{\frac{1}{M}} \end{aligned} \quad (8.2)$$

This is the mesonic $l = 0$ branch $\oplus_{r=1}^M \text{Sym}_p(\mathcal{C}_{r,l=0})$ with deformation parameter $\epsilon_{M,p}(r, l = 0)$ and low energy $SU(M)$ dynamics with scale $\Lambda(M, p, l = 0)$. This branch describes p D3-branes on the deformed conifold.

The other branches involve the strong coupling dynamics. The easiest way to find them is to dualize the $SU(N_1)$ theory as in [6]. This leads to an $SU(2N_2 - N_1 = p - M)$ gauge theory with dual quarks \hat{A} and \hat{B} , and the meson \mathcal{M} . Restoring the second gauge group $SU(N_2)$ we have a theory which is similar to the original one except that $p \rightarrow p - M$ and the superpotential is

$$\frac{1}{\mu} \hat{A} \mathcal{M} \hat{B} + h \mathcal{M} \mathcal{M} \quad (8.3)$$

(the parameter μ and its role are explained in [13].) For generic \mathcal{M} the dual quarks \hat{A} and \hat{B} acquire a mass and then the dual gauge group $SU(2N_2 - N_1 = p - M)$ can be integrated out leading back to (8.1). This way we recover the previously discussed mesonic branch with $l = 0$. However, this theory also has another branch which is not obvious in the original degrees of freedom.

We can integrate out \mathcal{M} in (8.3)

$$\mathcal{M} \sim \frac{1}{\mu h} \hat{B} \hat{A} \quad (8.4)$$

to find a theory which is similar to our original theory except that $p \rightarrow p - M$.⁵ This theory has several branches. One of them is a mesonic branch $\oplus_{r=1}^M \text{Sym}_{p-M}(\mathcal{C})$, whose deformation parameter will be determined shortly. This branch arises similarly to the

⁵ As explained in section 7 about the $p = M + 1$ theory, after we have done this we can no longer recover the $l = 0$ branch.

$l = 1$ branch for $p = M + 1$ where the role of the dual quarks was played by \mathcal{A} and \mathcal{B} . The $U(1)_{baryon}$ is again unbroken, and the classical limit of this branch coincides with a subspace of the $l = 0$ mesonic branch. We interpret it, as there, in terms of a bound state at threshold of $2M$ D5-branes and M anti-D5-branes. In addition, we find $p - M$ D3-branes moving on $\mathcal{C}_{r,l=1}$.

Now let us analyze the parameters of the low energy theory generalizing the discussion for $p = M + 1$ in section 7. For simplicity, we set the parameter μ in (8.3) equal to the scale of the group which is being dualized, Λ_1 . First, we write the gauge theory as

$$SU(\hat{N}_1 = p) \times SU(\hat{N}_2 = p - M) \quad (8.5)$$

to agree with our general notation with $N_1 \geq N_2$. Therefore we have

$$\hat{M} = M \quad ; \quad \hat{p} = p - M \quad (8.6)$$

Next, it is clear that the quartic coupling is

$$\hat{h} \sim \frac{1}{h\Lambda_1^2} \quad (8.7)$$

The instanton factor of the second group $SU(\hat{N}_2)$ is related to its dual $SU(N_1)$ using $\Lambda_1^{3M+p} \hat{\Lambda}_2^{p-3M} \sim \mu^{2p} = \Lambda_1^{2p}$ and therefore

$$\hat{\Lambda}_2 = \Lambda_1 \quad (8.8)$$

The instanton factor of the microscopic $SU(N_2 = p)$ is Λ_2^{p-2M} . After the duality this theory has $2(p - M)$ fundamental flavors and four adjoints. Therefore, its instanton factor is $\Lambda_L^{3p-2(p-M)-4p} = \Lambda_L^{2M-3p} \sim \Lambda_2^{p-2M} \Lambda_1^{4(M-p)}$. After the adjoints get a mass of order $h\Lambda_1^2$, the scale of the $SU(\hat{N}_1 = N_2)$ theory is

$$\hat{\Lambda}_1^{p+2M} \sim \Lambda_L^{2M-3p} (h\Lambda_1^2)^{4p} \sim h^{4p} \Lambda_1^{4(M+p)} \Lambda_2^{p-2M} \quad (8.9)$$

Now, we can use our earlier result about the $l = 0$ branch of this low energy theory. Using (8.6)-(8.9) we can express the results in terms of the original microscopic parameters

$$\begin{aligned} \hat{\Lambda}(\hat{M}, \hat{p}, l = 0)^{3\hat{M}} &\sim \hat{h}^{\hat{p}} \hat{\Lambda}_1^{3\hat{M}+\hat{p}} \sim (h\Lambda_1^2)^{M-p} (h^{4p} \Lambda_1^{4(M+p)} \Lambda_2^{p-2M}) \\ &= h^{M+3p} \Lambda_1^{2(3M+p)} \Lambda_2^{p-2M} \sim \Lambda(M, p, l = 1)^{3M} \end{aligned} \quad (8.10)$$

exactly as in (3.3).

Using (8.10) we can find the deformation parameter for $\hat{A}_{ai}\hat{B}^{bi}$ and finally for the eigenvalues of the meson $\sqrt{h}\mathcal{M}$ (use (8.4) with $\mu = \Lambda_1$)

$$\epsilon_{M,p}(r, l = 1) \sim \frac{h}{(h\Lambda_1)^2 \hat{h}} \hat{\epsilon}_{\hat{M}, \hat{p}}(r, l = 0) \sim \left(\hat{\Lambda}(\hat{M}, \hat{p}, l = 0)^{3M} \right)^{\frac{1}{M}} \sim (\Lambda(M, p, l = 1)^{3M})^{\frac{1}{M}} \quad (8.11)$$

which agrees with the general expression in (3.6).

Now it is clear how we can continue dualizing this way to find our final answer for the moduli space

$$\oplus_{r=1}^M \oplus_{l=0}^k \text{Sym}_{p-lM}(\mathcal{C}_{r,l}) \quad (8.12)$$

with

$$\begin{aligned} \epsilon_{M,p}(r, l) &\sim (\Lambda(M, p, l)^{3M})^{\frac{1}{M}} \\ \Lambda(M, p, l)^{3M} &\sim h^{p+l(M+2p)} \Lambda_1^{(3M+p)(l+1)} \Lambda_2^{(p-2M)l} \end{aligned} \quad (8.13)$$

After the duality, the integration out of the meson \mathcal{M} does not handle correctly the branch of the moduli space of the largest dimension ($l = 0$), but the next branch with $l = 1$ becomes manifest. This branch was interpreted as a bound state of $2M$ D5-branes and M anti-D5-branes near the tip of the conifold. As we continue to dualize the story repeats itself and we find more branches which correspond to bound states of $(l+1)M$ D5-branes and lM anti-D5-branes. The cascade stops when we use up all the available D5-branes in this fashion.

9. Consistency checks – relations between theories and branches

By considering various boundaries of the moduli space we can find relations between the different branches of different gauge theories. These relations provide nontrivial consistency checks of our expressions (3.3)(3.6).

First, consider the limit as one of the eigenvalues of \mathcal{M} is moved to infinity. In terms of the brane interpretation this corresponds to removing a D3-brane from the system. In the gauge theory this has the effect of changing $p \rightarrow p - 1$, and adding a $U(1)$ factor. In (4.9) we expressed the instanton factors of this low energy $SU(N_1 - 1) \times SU(N_2 - 1)$ theory in terms of the microscopic scales

$$\hat{\Lambda}_1^{3M+p-1} \sim \Lambda_1^{3M+p} h \quad ; \quad \hat{\Lambda}_2^{p-2M-1} \sim \Lambda_2^{p-2M} h \quad (9.1)$$

Clearly, the parameter h does not change, $\hat{h} = h$. It is easy to check that the transformations $p \rightarrow \hat{p} = p - 1$ and $\Lambda_{1,2} \rightarrow \hat{\Lambda}_{1,2}$ map

$$\begin{aligned}\hat{L}_1(M, \hat{p}) &= \hat{h}^{\hat{p}} \hat{\Lambda}_1^{3M+\hat{p}} \sim L_1(M, p) \\ \hat{L}_2(M, \hat{p}) &= \hat{h}^{M+\hat{p}} \hat{\Lambda}_2^{\hat{p}-2M} \sim L_2(M, p) \\ \hat{I}(M, \hat{p}) &= \hat{L}_1(M, \hat{p}) \hat{L}_2(\hat{M}, \hat{p}) = I(M, p)\end{aligned}\tag{9.2}$$

and therefore our relations (3.3)(3.6) are mapped consistently

$$\begin{aligned}\hat{\Lambda}(M, \hat{p}, l)^{3M} &\sim \hat{L}_1(M, \hat{p}) \hat{I}(M, \hat{p})^l \sim \Lambda(M, p, l)^{3M} \\ \hat{\epsilon}_{M, \hat{p}}(r, l) &\sim \left(\hat{\Lambda}(M, \hat{p}, l)^{3M} \right)^{\frac{1}{M}} \sim \epsilon_{M, p}(r, l)\end{aligned}\tag{9.3}$$

As expected, the moduli space of the remaining branes is not affected by removing a D3-brane.

Clearly, we can iterate this process as long as \tilde{p} remains positive. As explained in the previous sections, if we continue this way down to $\tilde{p} = 0$ we do not find the one complex dimensional baryonic branches. Instead, we find them as zero dimensional branches because effectively the baryon number symmetry is now gauged. As we continue down this road to smaller values of p , we loose the branches with large values of l . This is expected from the brane picture. However, it is important that the branches which are found have their correct deformations and scales.

The process of reducing p relates all $\Lambda(M, p, l)$ to $\Lambda(M, p = lM, l)$. Our second consistency check involves the value of this scale. When we discussed the $p = M$ theory in section 6, we checked that $\Lambda(M, p = M, l = 1)$ was obtained correctly in two different regions of the moduli space (6.3). This discussion is easily generalized to higher l . Far out along the baryonic branch of the $\tilde{p} = 0$ theory we have the expression (4.11) which is based on the nontrivial embedding of $SU(M)$. Near the origin of the moduli space the same expression is easily found using matching relations and the dual theory (we do not give the details here).

These two checks are nontrivial tests of our expressions (3.3)(3.6) and the brane interpretation. They reinforce the interpretation of the bound state being the same bound state for any value of p with the same M and l . Also, it is the same state which is visible semiclassically along the baryonic branch of the $\tilde{p} = 0$ theory. Clearly, removing D3-branes from the system should not affect the bound state.

Finally we comment that the duality transformations relate different values of l :

$$\hat{l} = l - 1 \quad (9.4)$$

More explicitly, we now generalize (8.10)(8.11). Every duality transformation maps the parameters as in (8.5)-(8.9)

$$\begin{aligned} \hat{M} &= M \\ \hat{p} &= p - M \\ \hat{h} &\sim \frac{1}{h\Lambda_1^2} \\ \hat{\Lambda}_2 &= \Lambda_1 \\ \hat{\Lambda}_1^{\hat{p}+2M} &\sim h^{4p} \Lambda_1^{4(M+p)} \Lambda_2^{p-2M} \end{aligned} \quad (9.5)$$

Therefore,

$$\begin{aligned} \hat{L}_1(M, \hat{p}) &= \hat{h}^{\hat{p}} \hat{\Lambda}_1^{3M+\hat{p}} \sim L_1(M, p) I(M, p) \\ \hat{L}_2(M, \hat{p}) &= \hat{h}^{M+\hat{p}} \hat{\Lambda}_2^{\hat{p}-2M} \sim \frac{1}{L_1(M, p)} \\ \hat{I}(M, \hat{p}) &= \hat{L}_1(M, \hat{p}) \hat{L}_2(\hat{M}, \hat{p}) = I(M, p) \end{aligned} \quad (9.6)$$

Then, using (9.4) we have

$$\begin{aligned} \hat{\Lambda}(\hat{M}, \hat{p}, \hat{l})^{3\hat{M}} &\sim \hat{L}_1(M, \hat{p}) \hat{I}(M, \hat{p})^{\hat{l}} \sim L_1(M, p) I(M, p)^l \sim \Lambda(M, p, l)^{3M} \\ \hat{\epsilon}_{\hat{M}, \hat{p}}(r, \hat{l}) &\sim \left(\hat{\Lambda}(\hat{M}, \hat{p}, \hat{l})^{3\hat{M}} \right)^{\frac{1}{\hat{M}}} \sim \epsilon_{M, p}(r, l) \end{aligned} \quad (9.7)$$

This last consistency check, which is associated with a change of l , has the following brane interpretation. As we cascade down the conifold we change the theory, $p \rightarrow p - M$, the parameters of the theory (9.5), and the branch $l \rightarrow l - 1$. This means that the number of D3-branes which are free to move is unchanged but the bound state includes fewer branes. Since this description is valid only closer to the tip of the conifold, this means that this bound state with smaller l is physically smaller. More heuristically, we can think of these bound states as being large atoms with many electrons. As we cascade down to smaller p and smaller l the space we look at is getting smaller and only electrons in inner shells fit in the space and can be included in the bound state.

The fact that the parameter $I(M, p)$ does not change under the transformations (9.5) is consistent with our identification $I(M, p) = e^{2\pi i \tau}$ in the type II theory. In the weak string coupling limit $|I(M, p)| \ll 1$ the deformation parameter

$$|\epsilon_{M, p}(r, l)| \sim |\epsilon_{M, p}(r, l = 0)| |I(M, p)|^{l/M} \sim |\epsilon_{M, p}(r, l = 0)| e^{-2\pi l / (g_s M)} . \quad (9.8)$$

This is in exact agreement with the dual string theory.⁶ We may embed the $SU(M + p) \times SU(p)$ gauge theory into a string compactification as in [25]. Then, the 5-form flux conservation gives the constraint $p = lM + N_{free}$. Here N_{free} is the number of mobile D3-branes, M is the number of units of the RR 3-form flux through the A-cycle, and l is the number of NS 3-form flux units through the B-cycle (i.e., the number of cascade steps). Since each cascade step reduces the mass-scale of the theory by a factor $e^{2\pi/(3g_s M)}$ [25,26], the string calculation gives $\epsilon \sim M_{string}^3 e^{-2\pi l/(g_s M)}$, in perfect agreement with (9.8).

10. Turning on a Fayet-Iliopoulos Term

In the previous sections we occasionally discussed the effects of gauging $U(1)_{baryon}$ and of turning on a Fayet-Iliopoulos parameter ξ . Since we will use such a term below, in this section we summarize and comment on these results.

Consider first the effect of gauging this symmetry with $\xi = 0$. The moduli space of vacua is the same as (3.1)

$$\oplus_{r=1}^M \oplus_{l=0}^k Sym_{p-lM}(\mathcal{C}_{r,l}) \quad (10.1)$$

except that for $p = kM$ the factor $Sym_0(\mathcal{C}_{r,l=k})$ is a point rather than a copy of \mathbb{C} . Also, the number of $U(1)$ factors in the low energy theory is always given by $p - lM$; i.e. all the moduli are in $\mathcal{N} = 4$ multiplets.

Now, let us consider a non-zero value for ξ . For a small $U(1)_{baryon}$ gauge coupling g , the effect of ξ can be analyzed in the low energy theory. For generic values of M and p , the low energy theory is $U(1)^{p-lM}$ and there are no light charged fields. ξ is the Fayet-Iliopoulos term of a particular linear combination of these $U(1)$ factors. Since there are no massless charged fields, it is clear that supersymmetry is broken, and for small g the vacuum energy is

$$V = \frac{g^2 \xi^2}{2} . \quad (10.2)$$

This agrees with our classical answer (4.6) but this derivation is more general because it includes also all the quantum corrections due to the strong $SU(N_1) \times SU(N_2)$ dynamics.

There is only one exception to this result. For $p = kM$ the theory with $\xi = 0$ has M isolated vacua with $l = k$ containing no low energy gauge fields. In these vacua the $U(1)_{baryon}$ gauge symmetry is Higgsed. Therefore, turning on ξ in these vacua does not

⁶ The following argument is due to J. Maldacena.

break supersymmetry, but instead it moves the vacuum in field space. It is important that even in this case of $p = kM$ the result (10.2) still applies to the other branches of the moduli space with $l = 0, \dots, k - 1$.

Let us consider the specific example of the theory with $p = kM + 1$ on the branch with $l = k$. Different values of k are related by the duality transformations (the example of cascading from $k = 1$ to $k = 0$ was discussed in section 7). Therefore, we can focus on the simplest case, $p = 1$, where we find an $SU(M + 1) \times U(1)_{baryon}$ gauge theory with the A and B fields. The moduli space is described by a single D3-brane moving on the deformed conifold $\mathcal{C}_{r,0}$: in the gauge theory its position is encoded in the meson fields $\mathcal{M}_{\alpha\dot{\alpha}}$. We will consider the leading order effect in the $U(1)_{baryon}$ gauge coupling g far out along the flat direction. There we can use the classical approximation to find the potential

$$\frac{1}{2}g^2 (|A_{\alpha\dot{\alpha}}|^2 - |B_{\dot{\alpha}}^a|^2 - \xi)^2 . \quad (10.3)$$

So, the potential of a moving D3-brane picks up a positive constant shift $g^2\xi^2/2$.

We can also see this effect without gauging $U(1)_{baryon}$. Consider the $p = M + 1$ theory and separate a single D3-brane. As we discussed at the end of section 4, here $SU(2M + 1) \times SU(M + 1)$ is broken to $SU(2M) \times SU(M) \times U(1)$. So, in addition to containing the meson $\mathcal{M}_{\alpha\dot{\alpha}}$, our low-energy theory is the $p = M$ theory with gauged $U(1)_{baryon}$, whose gauge coupling originates from the non-Abelian gauge coupling g_{YM} . This gauging removes the baryonic branch of this theory. But we could attempt to move the fields in the direction of the baryonic branch by turning on a nonzero value $\langle \mathcal{U} \rangle \sim U$ in (4.2). Since the baryonic branch is lifted, this breaks supersymmetry and leads to a potential for \mathcal{M} of order U^2 . In section 15, we will find the supergravity dual of this effect. We will treat this mobile D3-brane as a probe and will calculate its potential. The nonzero value for U will appear because we will place the D3-brane on a resolved warped deformed conifold.

11. Matching Gauge Theory and String Theory

In this section we would like to analyze various objects in the theory, and compare their gauge theory and string theory descriptions. These objects include domain walls, and confining and solitonic strings.

11.1. Domain Walls in the Gauge Theory

In field theories, domain walls interpolate between different vacua. What are the possible domain walls in the confining $SU(M+p) \times SU(p)$ gauge theory? First, it is clear that there are no domain walls interpolating between two different vacua on the same branch. Such a domain wall simply spreads out and becomes infinitely thick.

Second, we examine domain walls interpolating between vacua on different branches. The most interesting case is when the wall interpolates between two branches with different r but with the same value of l . Branches with the same l have a natural one-to-one map between them. Therefore, we consider a domain wall which interpolates between a point in branch r and its image in branch r' . As far as the low energy $SU(M)$ theory, this is a familiar situation of a domain wall which interpolates between two of the M vacua produced by the breaking of the \mathbb{Z}_{2M} R-symmetry to \mathbb{Z}_2 [14-16]. Therefore, we learn that this domain wall is BPS and its tension is

$$M \left| \Lambda(M, p, l)^3 (e^{\frac{2\pi i r}{M}} - e^{\frac{2\pi i r'}{M}}) \right| \quad (11.1)$$

For large M this becomes

$$M \left| \Lambda(M, p, l)^3 (e^{\frac{2\pi i r}{M}} - e^{\frac{2\pi i r'}{M}}) \right| \rightarrow 2\pi \left| \Lambda(M, p, l)^3 (r - r') \right| \quad (11.2)$$

Standard large M counting has $\Lambda(M, p, l)^3 \sim M$ [16], and the tension of the domain wall is of order M . Therefore, in the 't Hooft limit, this scales as a D-brane tension [16]. Indeed, in the string theory dual of our gauge theory these domain walls are the D5-branes wrapping the S^3 at the bottom of the deformed conifold $r - r'$ times [5,27,28].

The domain wall tension (11.1) is independent of the moduli. This is a general property of BPS domain walls. Consider a BPS domain wall which interpolates between a vacuum a and a vacuum b . Its tension is $|W(a) - W(b)|$, where $W(a)$ and $W(b)$ are the values of the superpotentials in the two vacua. Now assume that either the vacuum a , or the vacuum b or both are on a moduli space of supersymmetric vacua. Then, it is clear that $W(a)$ and $W(b)$ are independent of the moduli; otherwise, that superpotential would have led to a potential along the moduli space. Since $W(a)$ and $W(b)$ are independent of the moduli, so is their difference, which is the tension. This simple argument shows that the tension of a BPS domain wall is independent of the moduli.

A slightly more complicated example is obtained from the one we have just discussed by letting the domain wall interpolate between two vacua which are not isomorphic. It

is clear that the lowest energy configuration is obtained by first interpolating between two isomorphic points, as above, and then interpolating to the desired vacuum. It is also clear that this second step in the interpolation will make the wall spread out and make it non-BPS.

The most complicated example occurs when we try to interpolate between vacua with different values of l . Since there is no one-to-one correspondence between branches of the moduli space with different l , it is clear, by the argument above that such domain walls cannot be BPS. The most we can say about them is that their tension is bounded by the difference in the superpotential

$$\begin{aligned} T &> M \left| \Lambda(M, p, l)^3 e^{\frac{2\pi i r}{M}} - \Lambda(M, p, l')^3 e^{\frac{2\pi i r'}{M}} \right| \\ &= M \left| \Lambda(M, p, l = 0)^3 \left(I(M, p)^{\frac{l}{M}} e^{\frac{2\pi i r}{M}} - I(M, p)^{\frac{l'}{M}} e^{\frac{2\pi i r'}{M}} \right) \right| \\ &\rightarrow M \left| \Lambda(M, p, l = 0)^3 \left(e^{2\pi i \tilde{\tau} l} - e^{2\pi i \tilde{\tau} l'} \right) \right| \end{aligned} \quad (11.3)$$

where we have used $I = e^{2\pi i \tau} = e^{2\pi i \tilde{\tau} M}$ and the fact that in the 't Hooft limit, $\tilde{\tau}$ is of order one. In the supergravity approximation, $|\tilde{\tau}| \ll 1$. Then, we may expand

$$T > M \left| \Lambda(M, p, l = 0)^3 \left(e^{2\pi i \tilde{\tau} l} - e^{2\pi i \tilde{\tau} l'} \right) \right| \approx 2\pi M \left| \Lambda(M, p, l = 0)^3 \tilde{\tau} (l - l') \right| . \quad (11.4)$$

So, this tension is bounded from below by order M^2 . In the 't Hooft limit, this scales as an NS5-brane tension. Indeed, these domain walls are dual to the NS5-branes wrapping the \mathbb{S}^3 at the bottom of the deformed conifold [29].

11.2. Domain Walls in the Dual String Theory

In the supergravity duals, the BPS domain wall separating the adjacent vacua is a D5-brane wrapped over the round 3-sphere at $t = 0$ [5,27,28]. In section 14.1 we will show that the tension of this wrapped D5-brane does not depend on the baryonic branch modulus, in agreement with the field theory considerations. Therefore, to calculate the tension of the wrapped D5-brane, we will work at the \mathbb{Z}_2 symmetric locus on the baryonic branch, $|\mathcal{A}| = |\mathcal{B}|$, described by the warped deformed conifold solution [5]. Recall that the metric is

$$ds_{10}^2 = H_{KS}^{-1/2}(t) dx^2 + H_{KS}^{1/2}(t) ds_6^2 , \quad (11.5)$$

where ds_6^2 is the Calabi-Yau metric on the deformed conifold

$$\sum_{i=1}^4 z_i^2 = \varepsilon^2 . \quad (11.6)$$

Its explicit form is given, for example, in [5]. At $t = 0$ one finds a round 3-sphere of radius-squared $\varepsilon^{4/3}(2/3)^{1/3}$. Hence, its volume is $2\pi^2\varepsilon^2\sqrt{2/3}$. The tension of the domain wall is

$$T = \varepsilon^2 \frac{\sqrt{2/3}}{16\pi^3 g_s (\alpha')^3} . \quad (11.7)$$

Note that powers of $H_{KS}(0)$ cancel in this calculation, since the D5-brane has three directions within $\mathbb{R}^{3,1}$ and three within the deformed conifold.

To match the string and field theory parameters, we set (11.7) equal to the field theory result,

$$\Lambda(M, p, l)^3 \sim M \frac{\varepsilon^2}{g_s M (\alpha')^3} . \quad (11.8)$$

Since both ε and $g_s M$ are held fixed in the 't Hooft limit, we see that $\Lambda(M, p, l)^3$ is of order M [16].⁷ Thus, the IR scale kept fixed in the large M limit is

$$\tilde{\Lambda}(M, p, l) = M^{-1/3} \Lambda(M, p, l) , \quad (11.9)$$

and we find

$$\frac{\varepsilon^2}{(\alpha')^3} \sim g_s M \tilde{\Lambda}(M, p, l)^3 . \quad (11.10)$$

The cascading theory has another type of domain wall which separates vacua with adjacent values of l . The dimensions of the moduli space on the two sides of this domain wall are different, hence this domain wall cannot be BPS saturated. Therefore, its tension is not given by the difference between the values of the superpotential. In the supergravity dual this domain wall is an NS5-brane wrapped over the 3-sphere [29]. To see that this identification is correct, we note that the M units of RR flux through the 3-sphere require that M D3-branes end on the NS5-brane (this is the Hanany-Witten effect [30]). Hence, upon crossing the domain wall, we find M additional D3-branes corresponding to $l \rightarrow l - 1$. This is why the dimensions of the moduli space differ on the two sides of the domain wall. The presence of the M D3-branes attached to the wrapped NS5 makes it difficult to define the domain wall tension: it is a boundary term that must be separated from a much bigger bulk term related to the back-reaction of the M D3-branes filling $\mathbb{R}^{3,1}$ on the supergravity background. Hence, the calculation of the non-BPS domain wall tension is a difficult task.

⁷ Comparing with the conventions of (3.6), where $\epsilon \sim \Lambda(M, p, l)^3$, we find $\epsilon \sim M \frac{\varepsilon^2}{g_s M (\alpha')^3}$. The fact that in the 't Hooft limit $\epsilon \sim \Lambda^3$ scales as M , while ε is of order one can be traced back to the scaling of the coordinates \mathcal{M} by \sqrt{h} in (3.5). Since in the 't Hooft limit $h \sim M$, the deformation in terms of \mathcal{M} is of order one.

11.3. Interpolation in $g_s M$ and other comparisons

The large M cascading gauge theory has a continuous parameter $g_s M$. For small $g_s M$ the spacing between cascade steps is large [5], corresponding to the ‘choppy’ RG flow discussed in detail in [11]. Far in the IR such a theory is well-approximated by the usual $\mathcal{N} = 1$ supersymmetric gluodynamics; hence, we expect the square root of the confining string tension and the glueball masses to be of order $|\tilde{\Lambda}(M, p, l)|$. Let us compare these results with the supergravity predictions, which are valid for large $g_s M$ [26]:

$$T_s^{1/2} \sim \frac{\varepsilon^{2/3}}{\alpha' \sqrt{g_s M}} \sim \frac{|\tilde{\Lambda}(M, p, l)|}{(g_s M)^{1/6}}, \quad (11.11)$$

$$m_{glueball} \sim \frac{\varepsilon^{2/3}}{\alpha' g_s M} \sim \frac{|\tilde{\Lambda}(M, p, l)|}{(g_s M)^{2/3}}. \quad (11.12)$$

More generally, we have

$$T_s^{1/2} \sim |\tilde{\Lambda}(M, p, l)| f_s(g_s M), \quad m_{glueball} \sim |\tilde{\Lambda}(M, p, l)| f_g(g_s M), \quad (11.13)$$

where $f_s(g_s M)$ interpolates between a value of order one at small $g_s M$, and the $(g_s M)^{-1/6}$ fall-off at large values. Similarly, f_g interpolates between values of order one and the $(g_s M)^{-2/3}$ fall-off. Interpolations of this sort are typical in gauge/gravity dualities.

Now, let us discuss D-branes at the bottom of the warped deformed conifold. In the probe approximation, the D-string tension is

$$\frac{1}{2\pi\alpha' g_s H_{KS}(0)^{1/2}} \sim M \frac{|\tilde{\Lambda}(M, p, l)|^2}{(g_s M)^{4/3}}. \quad (11.14)$$

Note that it is proportional to M . In the non-compact conifold case, there is a problem with the probe approximation. The D-string introduces a monodromy of the massless pseudoscalar mode [8], which causes an IR logarithmic divergence. In the SUGRA calculation this divergence comes from the perturbation δF_{01r} introduced by the string stretched in the x^1 direction. Thus, to discuss the tension we have to introduce an IR cut-off. However, if we embed the deformed conifold in a string compactification, then the $U(1)_{baryon}$ symmetry is gauged, and the IR divergence is removed.

The D-string at the bottom of the warped deformed conifold should be dual to a solitonic string in the cascading gauge theory, which couples to the Goldstone boson [8].

The field theory discussion of such solitonic strings again presumes either an IR regulator, which removes the logarithmic divergence in the tension, or a gauging of $U(1)_{baryon}$ which turns the string into a string of Abrikosov-Nielsen-Olesen type. On general grounds, the tension of this string should satisfy

$$T_{soliton} = M \left| \tilde{\Lambda}(M, p, l) \right|^2 f_{soliton}^2(g_s M) , \quad (11.15)$$

where $f_{soliton}(g_s M)$ falls off as $(g_s M)^{-2/3}$ at infinity. Note that there is no such soliton in the $\mathcal{N} = 1$ supersymmetric $SU(M)$ gauge theory. Therefore, we expect $f_{soliton}$ to diverge as $g_s M \rightarrow 0$.

Another very interesting non-BPS object is the anti-D3 brane, whose tension is

$$\frac{1}{8\pi^3(\alpha')^2 g_s H_{KS}(0)} \sim M \frac{\left| \tilde{\Lambda}(M, p, l) \right|^4}{(g_s M)^{5/3}} . \quad (11.16)$$

At a general coupling, we expect the energy of this excitation per unit volume to behave as

$$M \left| \tilde{\Lambda}(M, p, l) \right|^4 f_D^4(g_s M) . \quad (11.17)$$

Again, this object might not be present in the pure supersymmetric gluodynamics; therefore, $f_D(g_s M)$ should blow up near zero. Thus, this is a very heavy object in the limit of widely spaced cascade steps. The smallest theory where it may exist is $SU(2M) \times SU(M)$ that appears at the bottom of the cascade. The fact that the tension scales as M suggests that only one eigenvalue of the meson matrix \mathcal{M} is excited.

12. Supergravity Dual of the Cascading Theory on the Baryonic Branch

In this and the subsequent sections we review the dual supergravity description of the baryonic branch of the cascading $SU((k+1)M) \times SU(kM)$ gauge theory, and compare various supergravity observables with the gauge theory along this branch. The simplest gauge theory picture of the baryonic branch is found in the far infrared $SU(2M) \times SU(M)$ theory where

$$\mathcal{A} = i\Lambda_1^{2M} \zeta , \quad \mathcal{B} = i\Lambda_1^{2M} / \zeta , \quad (12.1)$$

and ζ is the complex modulus for the branch. The gauge theory with $|\zeta| = 1$ is described by the warped deformed conifold solution of [5]. The gauge theory has a pseudoscalar Goldstone mode [7] corresponding to changes in the phase of ζ . Its supergravity dual was

constructed in [8]. This mode vanishes at zero momentum, in agreement with the fact that the Goldstone boson has only derivative couplings. Therefore, position-independent changes of the phase of ζ do not produce any new supergravity backgrounds.

The baryonic branch of the supergravity backgrounds is labelled by a real parameter $|\zeta|$. In [8] it was proposed that this branch falls within the Papadopoulos-Tseytlin (PT) ansatz [9] for backgrounds of IIB SUGRA describing the deformed conifold with fluxes. The ten-dimensional metric of the PT ansatz is⁸

$$ds_{10}^2 = H^{-1/2} dx_m dx_m + e^x ds_6^2, \quad (12.2)$$

$$ds_6^2 = (e^g + a^2 e^{-g})(e_1^2 + e_2^2) + e^{-g} \sum_{i=1}^2 (\epsilon_i^2 - 2a e_i \epsilon_i) + v^{-1}(\tilde{\epsilon}_3^2 + dt^2),$$

where H, x, g, a, v are functions of the radial variable t . The definitions of the 1-forms, and the ansatz for H_3, F_3, F_5 are reviewed in Appendix A; we ask the reader to refer to the notation there. While the necessary backgrounds are quite complicated, they simplify considerably in the large radius (UV) limit, where they approach the asymptotic cascade form found in [4]. This asymptotic may be approximated by $AdS_5 \times T^{1,1}$ modulo slowly-varying logarithms [4] which are present due to the logarithmic RG flow in the dual gauge theory [3].

The PT ansatz is $SU(2) \times SU(2)$ invariant but in general breaks the \mathbb{Z}_2 symmetry that interchanges the two \mathbb{S}^2 's of $T^{1,1}$. In the field theory the corresponding symmetry is the interchange of A_α with B_α accompanied by charge conjugation in both $SU(N_1)$ and $SU(N_2)$ [1]. This \mathbb{Z}_2 symmetry is restored for the warped deformed conifold solution of [5] corresponding to $|\zeta| = 1$. Since the breaking of this discrete symmetry is associated with the resolution of the conifold, the solutions with broken \mathbb{Z}_2 may be called *resolved warped deformed conifolds*.

The PT ansatz was originally introduced in search of an extrapolation between the warped deformed conifold (KS) background [5], which preserves the \mathbb{Z}_2 symmetry, and the Maldacena-Nunez (MN) background [27] which breaks it. In [8] the linearized deformations around the KS background, which are \mathbb{Z}_2 odd, were found using the PT ansatz. They were interpreted as the supergravity duals of small motions along the baryonic branch of the cascading gauge theory, corresponding to $|\zeta| \approx 1$. It has been conjectured that far along the baryonic branch the background approaches the MN background [31]. However, this

⁸ Following [10], we use this ansatz for the string frame metric.

cannot be true far in the UV since the MN background asymptotes to a linear dilaton rather than to the KT solution [4]. Subsequently, Butti, Graña, Minasian, Petrini and Zaffaroni (BGMPZ) wrote a remarkable paper [10], where the method of $SU(3)$ structures was used to derive a system of coupled first-order equations for the functions $a(t)$ and $v(t)$, describing an $\mathcal{N} = 1$ supersymmetric solution to the PT ansatz. The solution of these equations determines other unknown functions (see Appendix B), so that the problem of constructing the family of supergravity duals of the entire baryonic branch became tractable, at least numerically. It turns out that the backgrounds far along the baryonic branch do approach the appropriately shifted MN solution *in the IR*, yet in the UV they have the cascade asymptotics of [4].

12.1. Relation between the warp factor and the dilaton

We will be particularly interested in the the dilaton profile $\phi(t)$ and the warp factor $H(t)$ which determine the tensions of many probe branes. In our conventions, the position-dependent string coupling is $g_s e^{\phi(t)}$, and we set $\phi(\infty) = 0$. The dilaton profile is determined by [10]

$$\phi' = \frac{(C-b)(aC-1)^2}{(bC-1)S} e^{-2g} , \quad (12.3)$$

with the definitions of functions b, C, S given in Appendix B. The equation for the warp factor may be written in the form

$$H' = -K(t)e^{-2x(t)}H(t) , \quad (12.4)$$

which implies that the self-dual 5-form field strength is

$$g_s F_5 = d(H^{-1}) \wedge d^4x + K(t)e_1 \wedge e_2 \wedge \epsilon_1 \wedge \epsilon_2 \wedge \epsilon_3 , \quad (12.5)$$

i.e. $g_s C_{0123} = H^{-1}(t)$. Using (12.3) and formulae in Appendix B, we find that

$$K(t)e^{-2x(t)} = \frac{2\phi'}{1 - e^{2\phi(t)}} .$$

Hence, (12.4) may be written in the form

$$H' = -\frac{2\phi'}{1 - e^{2\phi(t)}} H(t) . \quad (12.6)$$

This may be integrated to give

$$H(t) = \tilde{H} \left(e^{-2\phi(t)} - 1 \right) , \quad (12.7)$$

where \tilde{H} is an integration constant. To achieve a decoupled field theory in gauge/gravity dualities, one requires that the warp factor $H(t)$ vanishes at infinity. Since $\phi(\infty) = 0$, (12.7) clearly satisfies this requirement for any \tilde{H} . A more detailed analysis of the boundary conditions at large t , which will allow us to determine \tilde{H} , will be presented in the next section.

13. Boundary Conditions and Analysis of Solutions

To specify the solution completely, we need to fix the boundary conditions in the UV region $t \rightarrow \infty$. In order to find the correct boundary conditions on the solutions along the baryonic branch, let us recall the \mathbb{Z}_2 symmetric KS solution [5]. In terms of the PT variables, this solution has

$$\begin{aligned} a_{KS} &= -\frac{1}{\cosh(t)} , \\ v_{KS} &= \frac{3}{2} \left(\coth(t) - \frac{t}{\sinh^2(t)} \right) , \\ e^{g_{KS}} &= \tanh t , \\ \phi_{KS} &= 0 , \\ e^{-4A_{KS}(t)} &= H_{KS}(t) = 2^{-8/3} \gamma I(t) , \end{aligned} \tag{13.1}$$

where we define

$$I(t) = \int_t^\infty dx \frac{x \coth x - 1}{\sinh^2 x} (\sinh 2x - 2x)^{1/3} , \quad \gamma = 2^{10/3} (g_s M \alpha')^2 \varepsilon^{-8/3} . \tag{13.2}$$

One finds [26] that $I(0) \approx 0.71805$, while for large t ,

$$I(t) \rightarrow 3 \cdot 2^{-7/3} (4t - 1) e^{-4t/3} + \dots$$

The large t expansion of the warp factor is therefore given by

$$\gamma^{-1} H(t) = \frac{3}{32} e^{-4t/3} (4t - 1) - \frac{25t^2 - 85t + 12}{125} e^{-10t/3} + O\left(e^{-16t/3}\right) . \tag{13.3}$$

Moving along the baryonic branch away from the \mathbb{Z}_2 symmetric solution of [5] corresponds to changing expectation values of fields in the cascading gauge theory. In the dual supergravity description, such changes typically preserve the leading asymptotics of the fields but affect the sub-leading terms [32]. This is the standard fact for asymptotically AdS spaces, and is expected to apply also to the cascading case where the UV asymptotics differ from AdS only by logarithmic corrections. Thus, for the entire baryonic branch of solutions we will require that the leading asymptotics are the same as in the KS case, i.e. $a(t) \rightarrow -2e^{-t}$, $\gamma^{-1} H(t) \rightarrow \frac{3}{32} e^{-4t/3} (4t - 1)$, etc. Similarly, we require that $\phi(\infty) = 0$.

13.1. Expansion around the KS solution

The baryonic branch solutions that break the \mathbb{Z}_2 symmetry slightly were found in [8]:

$$a(t) = a_{KS}(t)(1 - 2^{-5/3}UZ(t)) + O(U^2), \quad e^g = e^{g_{KS}}(1 - 2^{-5/3}UZ(t)) + O(U^2), \quad (13.4)$$

where

$$Z(t) = \frac{\tanh t - t}{(\cosh t \sinh t - t)^{1/3}}. \quad (13.5)$$

Thus, the asymptotic expansion of $a(t)$ is⁹

$$a(t) = -2e^{-t} + Ue^{-5t/3}(-t + 1) + \dots \quad (13.6)$$

and U parameterizes the resolution of the conifold [32], which breaks the \mathbb{Z}_2 symmetry. In the gauge theory this parameter is proportional to the expectation value of the \mathbb{Z}_2 odd operator \mathcal{U} (4.2). The corresponding metric component measures the difference between the radii-squared of the (e_1, e_2) two-sphere and the (ϵ_1, ϵ_2) two-sphere:

$$e^x (e^g + e^{-g}(a^2 - 1)) \sim U(t \coth t - 1)H_{KS}^{1/2}(t) + O(U^2). \quad (13.7)$$

For large t , this falls off as $t^{3/2}e^{-2t/3} \sim \varepsilon^{4/3}(\ln r)^{3/2}r^{-2}$ (here $r \sim \varepsilon^{2/3}e^{t/3}$ is the radial variable of the near-AdS asymptotic [4]). This is in agreement with \mathcal{U} having dimension 2 [32]. Hence, the expectation value of the operator may be read off from the coefficient of the leading asymptotic (see [33] for a study of one-point functions in the cascading background):

$$\langle \mathcal{U} \rangle \sim MU \frac{\varepsilon^{4/3}}{(\alpha')^2}. \quad (13.8)$$

The expectation value of \mathcal{U} is related to ζ (see (12.1)) through

$$\langle \mathcal{U} \rangle \sim M\Lambda_1^2 \ln |\zeta|. \quad (13.9)$$

Therefore,

$$U \sim \ln |\zeta|. \quad (13.10)$$

⁹ The parameter U coincides with a_{UV} introduced in [10]. We use U here rather than a_{UV} to stress the fact that U is proportional to the expectation value of the operator \mathcal{U} (4.2). Therefore, U parameterizes the IR physics rather than UV: it is the modulus of the vacuum on the baryonic branch.

As shown in [10], the UV asymptotic expansions of ϕ and H are

$$\phi(t) = -\frac{3}{64}U^2e^{-4t/3}(4t-1) + O\left(U^4e^{-8t/3}\right) , \quad (13.11)$$

$$\gamma^{-1}H(t) = \frac{3}{32}e^{-4t/3}(4t-1) - \frac{3}{32 \cdot 512}U^2(256t^3 - 864t^2 + 1752t - 847)e^{-8t/3} + O\left(e^{-10t/3}\right) \quad (13.12)$$

Comparing them with (12.7), we find that

$$\tilde{H} = \gamma U^{-2} . \quad (13.13)$$

In fact, taking U to zero in (12.7), we find the expression

$$\phi(t) = -2^{-11/3}U^2I(t) + O\left(U^4\right) \quad (13.14)$$

valid for all t ; i.e. the $O(U^2)$ term in $\phi(t)$ is proportional to $H_{KS}(t)$. The UV expansion (13.11) is reproduced by (13.3).

Moving along the baryonic branch corresponds to changing $U \sim \ln|\zeta|$. It is also useful to parameterize the branch using the parameter $y = y(U)$ which is defined via the IR expansion [10]

$$a = -1 + \left(\frac{1}{2} + \frac{y}{3}\right)t^2 + \dots \quad (13.15)$$

Comparing with the notation in the BGMPZ paper [10],

$$y = 3\xi_{BGMPZ} - 3/2 , \quad (13.16)$$

but we will reserve ξ for denoting the Fayet-Iliopoulos term. Now, $y \in (-1, 1)$ and the \mathbb{Z}_2 simply acts as $y \rightarrow -y$. Thus, at the KS point $U = y = 0$, corresponding to $|\mathcal{A}| = |\mathcal{B}|$. As $|\mathcal{A}| \rightarrow 0$, we instead have $y \rightarrow -1$ and $U \rightarrow -\infty$; in this limit the MN solution is approached in the IR, but the UV boundary conditions correspond to the cascade rather than the linear dilaton. y may be determined as a function of U through numerical integration.

13.2. Behavior far along the baryonic branch

The initial idea motivating the PT ansatz was that it may interpolate between the KS and the MN solutions. For the MN solution corresponding to $y = -1$ or $U \rightarrow -\infty$,

$$a_{MN} = -\frac{t}{\sinh(t)} , \quad v_{MN} = \sqrt{-1 + 2t \coth t - \frac{t^2}{\sinh^2 t}} . \quad (13.17)$$

We see that this does not have the asymptotics (13.6), which indicates that $y = -1$ is a singular point which has to be excluded from the baryonic branch. However, the solution can be arbitrarily close to this point and still lie on the baryonic branch. In fact, far along the baryonic branch the solutions become close to the MN solution in the IR, but strongly depart from it in the UV: in the UV all baryonic branch solutions have the “cascading” KT asymptotics that are $AdS_5 \times T^{1,1}$ modulo slowly varying logarithms, while the MN solution asymptotes to the linearly rising dilaton:

$$e^{2\phi_{MN}} \sim \sinh t \left(-1 + 2t \coth t - \frac{t^2}{\sinh^2 t} \right)^{-1/2} . \quad (13.18)$$

For the solution on the baryonic branch, we instead fix $\phi(\infty) = 0$. Then the IR value of the dilaton field, $\phi(t=0) = \phi_0$, starts from 0 for $U = 0$ and approaches $-\infty$ for $|U| \rightarrow \infty$. Therefore, ϕ_0 and U are both zero in the KS case and approach minus infinity in the MN limit $y \rightarrow -1$. We will later show that for large $|U|$, $e^{-\phi_0} \sim |U|^{3/4}$; i.e. the effective string coupling is much weaker in the IR than in the UV.¹⁰ This means that for $|U|$ so large that $g_s M e^{\phi_0}$ becomes small, the supergravity background becomes highly curved in the IR and cannot be trusted. This follows from the fact that the radius-squared of the S^3 at $t = 0$ is of order $\alpha' g_s M e^{\phi_0}$.

In Figures 1 and 2 we present the plots of $a(t)$, $v(t)$ and of $\phi(t)$, $H(t)$ for the KS ($y = 0$), MN ($y = -1$) and intermediate values $y = -3/4$ ($U \approx -3.3$) and $y = -0.99$ ($U \approx -20.1$).

Figure 3 contains the profile of the dilaton for $y = -0.99$ and the exact MN dilaton shifted in a way that it starts from the same value at $t = 0$ as $\phi_{y=-0.99}$. The two graphs are almost identical near $t = 0$ but $\phi_{y=-0.99}$ quickly approaches zero while ϕ_{MN} grows without bound. The plots of $\phi(t)$, $a(t)$ and $v(t)$ show that even the $y = -0.99$ solution approximates the MN solution well only for t up to around 2. More generally, one can argue that, as $y \rightarrow -1$, the solutions approximate the MN solution up to $t \sim -\ln(1+y)$. Thus, the approach of the IR behavior to that of the MN solution as $y \rightarrow -1$ is logarithmically slow.

¹⁰ If instead of keeping g_s fixed, we take a double scaling limit where $U \rightarrow -\infty$, and $g_s \sim |U|^{3/4}$, then we recover the MN solution [10].

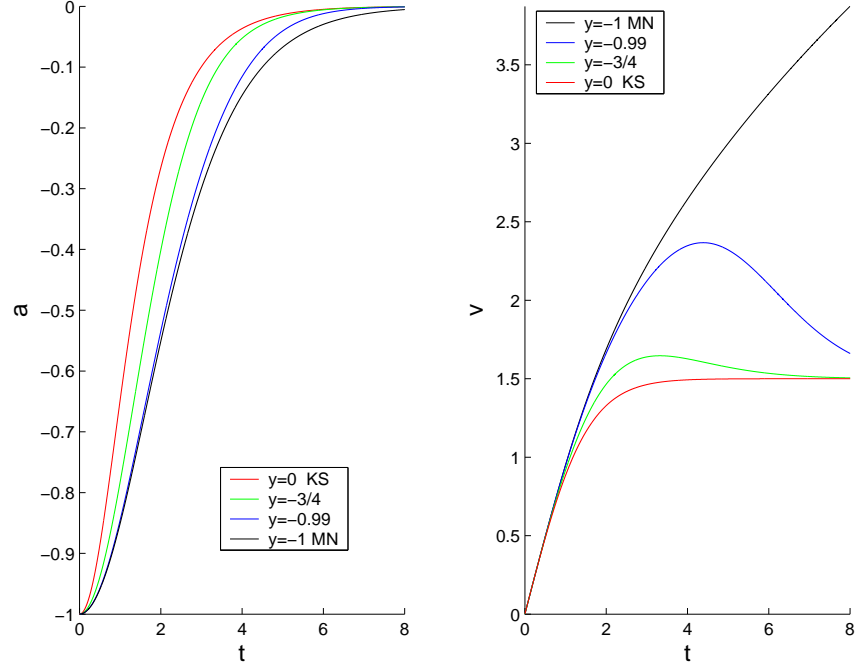


Fig. 1: Plots of $a(t)$ and $v(t)$. The KS plot is shown in red, $y = -3/4$ ($U \approx -3.3$) in green, $y = -0.99$ ($U \approx -20.1$) in blue, and MN ($y = -1$) in black.

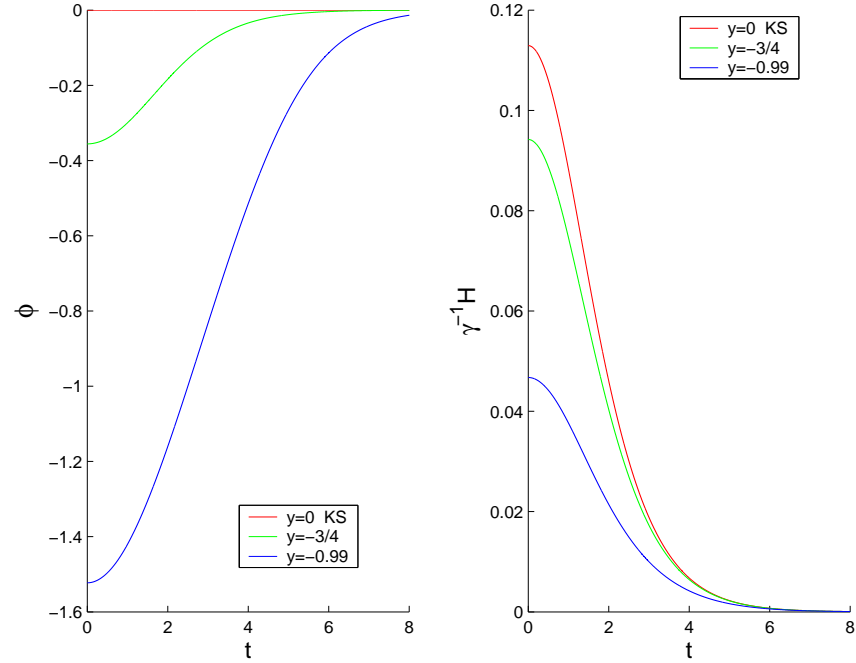


Fig. 2: Plots of $\phi(t)$ and $H(t)$. The KS plot is shown in red, $y = -3/4$ ($U \approx -3.3$) in green, $y = -0.99$ ($U \approx -20.1$) in blue.

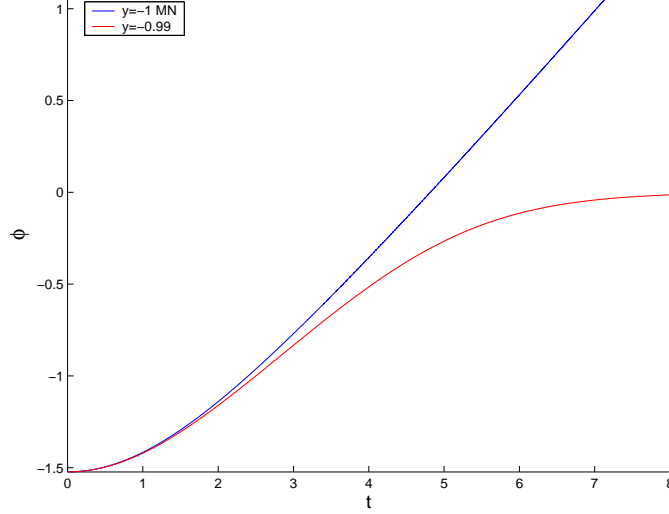


Fig. 3: The red line is the dilaton profile for $y = -0.99$ ($U \approx -20.1$). The blue line is the MN dilaton profile shifted in a way that it starts from the same value at $t = 0$ as $\phi_{y=-0.99}$.

14. The IR physics

The IR physics is governed by the geometry near the origin $t \rightarrow 0$:

$$\begin{aligned}
 ds_{10}^2 &= H_0^{-1/2} dx^2 + \frac{e^{\phi_0} \lambda}{2} (dt^2 + g_5^2 + 2g_3^2 + 2g_4^2) + O(t^2) , \\
 g_5 &= \tilde{\epsilon}_3, \quad g_3 = \frac{e_1 + \epsilon_3}{\sqrt{2}}, \quad g_4 = \frac{e_2 + \epsilon_4}{\sqrt{2}} , \\
 H_0 &= \gamma U^{-2} (e^{-2\phi_0} - 1), \quad \lambda^2 = y^{-2} (1 - e^{2\phi_0}) .
 \end{aligned} \tag{14.1}$$

We see that in the far IR region the geometry is just $\mathbb{R}^{3,1} \times \mathbb{S}^3 \times \mathbb{R}^3$. The radius-squared of \mathbb{S}^3 is $R^2 = e^{\phi_0} \lambda$.

14.1. Tension of the BPS Domain Wall

A D5-brane wrapped over the round \mathbb{S}^3 at $t = 0$ is the BPS domain wall separating two adjacent vacua (there are M inequivalent vacua corresponding to the phase of the gluino condensate). This is well-known to be a BPS object in the gauge theory, and we will see a reflection of this in the dual string theory: the tension does not depend on the baryonic branch parameter U .

The tension of the wrapped D5-brane is

$$T = \frac{1}{2(2\pi\alpha')^3 g_s} H_0^{-3/4} e^{\phi_0/2} \lambda^{3/2} = \frac{1}{2(2\pi\alpha')^3 g_s} H_0^{-3/4} e^{\phi_0/2} |y|^{-3/2} (1 - e^{2\phi_0})^{3/4} . \tag{14.2}$$

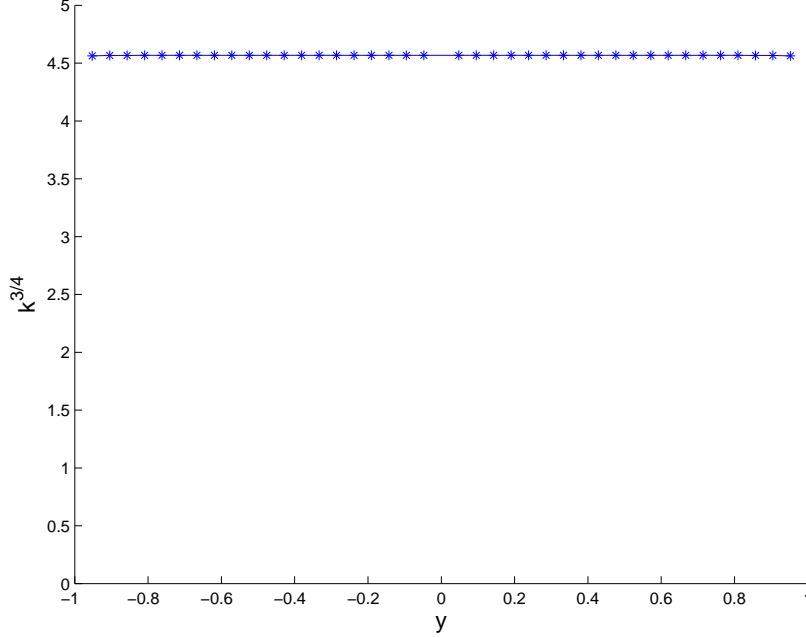


Fig. 4: The tension of a wrapped D5 brane

A numerical plot of this quantity as a function of U is given in Figure 4. It is constant within the numerical precision of the calculation. This is a nice check of the boundary conditions we have imposed on the supergravity solution.

Let us introduce k via

$$2g_s(2\pi\alpha')^3T = \gamma^{-3/4}k^{3/4} = H_0^{-3/4}e^{\phi_0/2}\lambda^{3/2} . \quad (14.3)$$

The value of k is easy to find at the KS point where, according to [5,26]

$$\lambda_{KS} = (kH_{KS}(0))^{1/2} = 6^{-1/3}2I(0)^{1/2} = 0.93266 , \quad (14.4)$$

and therefore

$$k = 2^4 3^{-2/3} . \quad (14.5)$$

The irrational constant $I(0)$ cancels because at the KS point all dependence on $H(0)$ cancels for such a wrapped brane: it has 3 directions within the conifold and 3 directions within $\mathbb{R}^{3,1}$. This is indicative of the BPS nature of the wrapped D5-brane.

The constancy of k provides us with a relation between U and the quantities ϕ_0 and y which are determined through integrating the equations from large t to $t = 0$:

$$U^2 = ky^2 e^{-8/3\phi_0} . \quad (14.6)$$

At large $|U|$, $|y|$ approaches 1. Hence, using (14.6), we see that $e^{-\phi_0}$ scales as $|U|^{3/4}$. Using (14.6), we also find

$$H_0 = \gamma \frac{e^{-2\phi_0} - 1}{U^2} = \frac{y^{-2}\gamma}{k} e^{2\phi_0/3} (1 - e^{2\phi_0}) . \quad (14.7)$$

This implies that $H_0 \sim |U|^{-1/2}$ for large $|U|$.

14.2. Tensions of the fundamental string and anti-D3 brane

The dual of the confining string is the fundamental string placed at $t = 0$. As follows from (14.1), its tension is

$$T_s = \frac{1}{2\pi\alpha'} H_0^{-1/2} . \quad (14.8)$$

This is not constant along the branch, in agreement with the fact that the confining string is not BPS saturated. Using (14.3), we have

$$H_0^{-1/2} = \gamma^{-1/2} k^{1/2} e^{-\phi_0/3} \lambda^{-2} . \quad (14.9)$$

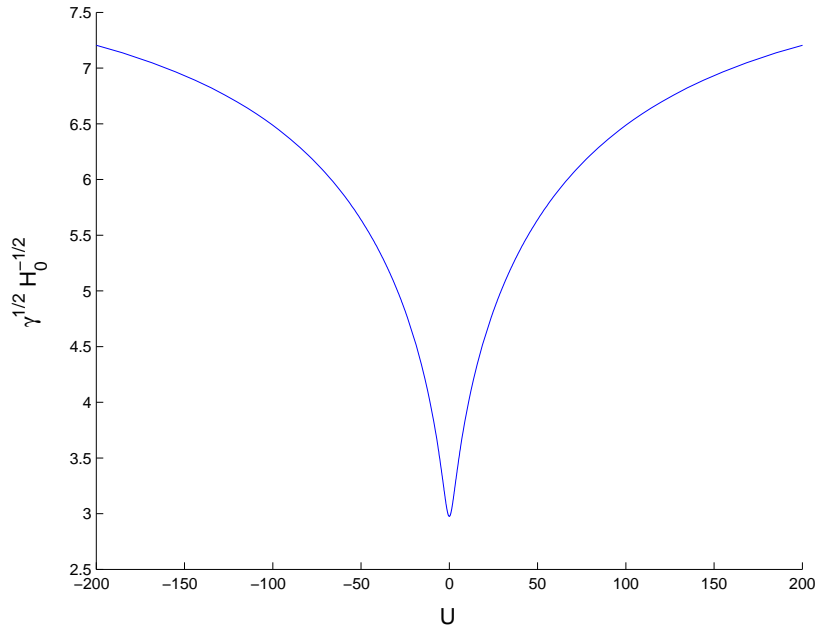


Fig. 5: The confining string tension

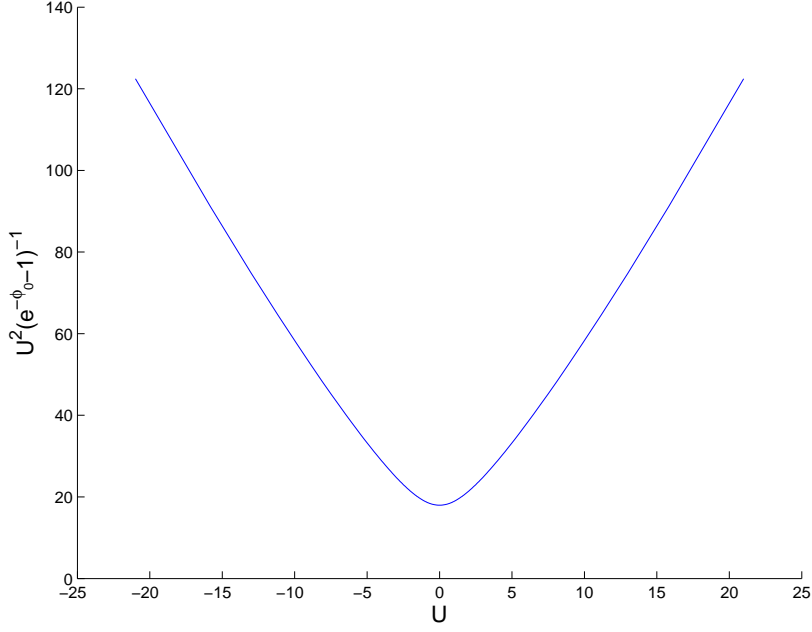


Fig. 6: The anti-D3 brane tension

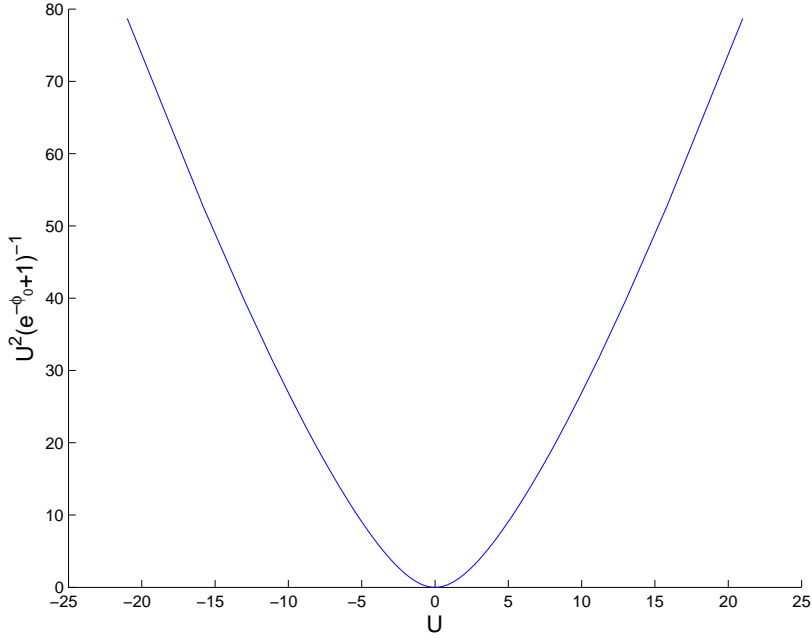


Fig. 7: The D3 brane tension

At large $|U|$, λ approaches 1; hence, T_s diverges as $e^{-\phi_0/3} \sim |U|^{1/4}$. Figure 5 shows T_s as a function of U . We have also calculated some glueball masses along the baryonic branch, and we find that they again diverge as a positive power of $|U|$. We postpone a detailed presentation of the glueball results to a future publication.

Now consider an anti-D3-brane parallel to $\mathbb{R}^{3,1}$.¹¹ It falls to $t = 0$ for all values of U including $U = 0$. The tension of an anti-D3 brane placed at $t = 0$ is

$$T_{\overline{D3}} = T_3 H_0^{-1} (e^{-\phi_0} + 1) = \frac{T_3}{\gamma} \frac{U^2}{e^{-\phi_0} - 1} , \quad (14.10)$$

where the normalization factor is the D3-brane tension

$$T_3 = \frac{1}{8\pi^3 (\alpha')^2 g_s} . \quad (14.11)$$

The plot of this quantity as a function of U is shown in Figure 6. For large $|U|$ it again grows as $|U|^{5/4}$. The tension does not vanish at $U = 0$ reflecting the fact that the anti-D3 brane breaks supersymmetry in the KS background; furthermore, for small U it rises as $\sim U^2$. This means that the scalar mode corresponding to motion along the baryonic branch has become massive. Thus, the non-supersymmetric metastable state of the gauge theory, which is dual to the anti-D3 brane at the bottom of the KS solution, does not have a baryonic branch. For consistency, the massless pseudoscalar Goldstone mode should also be absent from the spectrum. In fact, it is eaten by the $U(1)$ world volume gauge field on the anti-D3 brane, which becomes massive. The term in the world volume gauge theory responsible for this is

$$\int dA \wedge C_2 = - \int A \wedge F_3 . \quad (14.12)$$

Since $F_3 \sim *(da)$ [8], where a is the Goldstone mode, (14.12) becomes

$$\int A^\mu \partial_\mu a , \quad (14.13)$$

which leads to the Higgs mechanism for the world volume $U(1)$.

15. The D3-brane and a new Approach to Brane Inflation

The situation is even more interesting for a D3-brane parallel to $\mathbb{R}^{3,1}$. Now the relevant cascading gauge theory is $SU(1 + M(k + 1)) \times SU(1 + Mk)$. A detailed discussion of the $k = 0$ theory was given in sections 5 and 10, and of the $k = 1$ theory in sections 7 and 10. We will find that the dual string theory results are in remarkable agreement with the gauge theory.

¹¹ We thank J. Maldacena for his very useful input on the following paragraph.

The potential of the D3-brane is

$$V(t) = T_3 H^{-1}(t) (e^{-\phi(t)} - 1) . \quad (15.1)$$

The first term comes from the Born-Infeld term and has a factor of $e^{-\phi(t)}$; the second term, originating from the interaction with the background 4-form C_{0123} , does not have this factor. For the KS solution ($U = 0$), $\phi(t) = 0$ and $V(t) = 0$; therefore, the potential vanishes and the D3-brane may be located at any point on the deformed conifold. For $U \neq 0$ we may use (12.7) and (13.13) to write

$$V(t) = \frac{T_3}{\gamma} \frac{U^2}{e^{-\phi(t)} + 1} . \quad (15.2)$$

Since $\phi(t)$ is a monotonically increasing function, the D3-brane is attracted to $t = 0$.

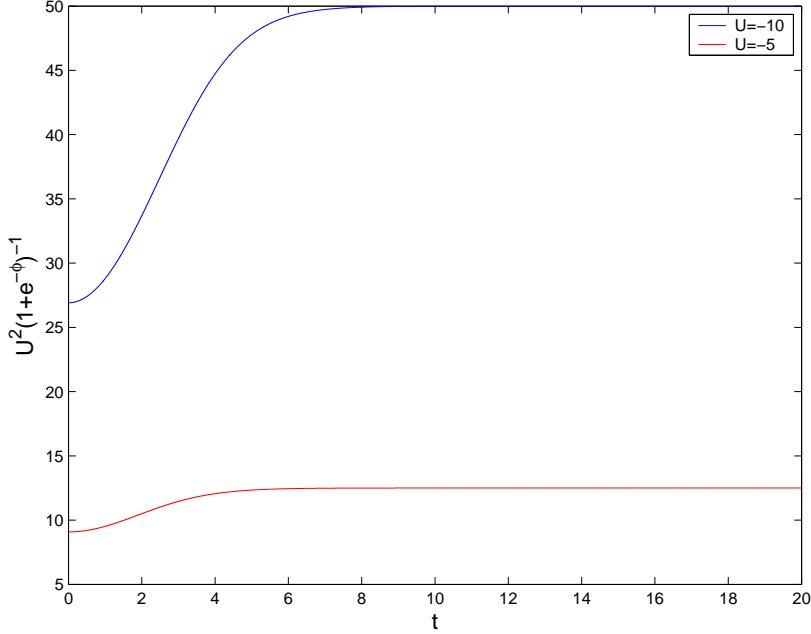


Fig. 8: Plots of the D3-brane potential as a function of t for $U = -5$ and $U = -10$.

Plots of the potential (15.1) for $U = -5$ and $U = -10$ are shown in Figure 8. Note that even at $t = 0$ the D3-brane has a finite tension and breaks the supersymmetry.

The fact that the D3-brane has a non-vanishing potential for a background with $U \neq 0$ follows from the explicit form of the 10-dimensional Killing spinor [10]¹²

$$\begin{aligned} \Psi &= \alpha \psi + \beta \psi^* , \\ \alpha &= \frac{e^{\phi/8} (1 + e^\phi)^{3/8}}{(1 - e^\phi)^{1/8}} , \quad \beta = i \frac{e^{\phi/8} (1 - e^\phi)^{3/8}}{(1 + e^\phi)^{1/8}} . \end{aligned} \quad (15.3)$$

¹² We thank I. Bena for pointing this out to us.

The spinor ψ has a definite 4-dimensional chirality, and its charge conjugate ψ^* has the opposite chirality. At the KS point $\beta = 0$, and Ψ has a definite 4-dimensional chirality. In this case a D3-brane is a BPS state. But for $U \neq 0$ Ψ does not have a definite four-dimensional chirality, so none of the supersymmetries of the background are preserved by the D3-brane.

For small U we may expand

$$V(t) = \frac{T_3}{8\gamma} \left(4U^2 - 2^{-8/3} I(t) U^4 + O(U^6) \right) . \quad (15.4)$$

Hence, the attractive force on the D3-brane appears only at order U^4 . Note that in the DBI action the kinetic term for the radial variable t does not have a canonical form $\sim \dot{t}^2$, where $\dot{t} = \partial t / \partial x^0$. Instead, we find the action

$$T_3 \left(f^2(t) \dot{t}^2 + H^{-1}(t) (e^{-\phi(t)} - 1) \right) , \quad (15.5)$$

where

$$f^2 = \frac{e^{-\phi+x}}{H^{1/2} v} . \quad (15.6)$$

The radial variable q that has the canonical kinetic term may be found by solving the equation $dq/dt = f(t)\sqrt{T_3}$. In the asymptotic KT region, q coincides with $\sqrt{T_3}r$ where the standard variable $r \sim \varepsilon^{2/3} e^{t/3}$.

In models of inflation, one typically defines the parameters¹³

$$\epsilon = \frac{M_{Pl}^2}{2} V^{-2} \left(\frac{\partial V}{\partial q} \right)^2 , \quad \eta = M_{Pl}^2 V^{-1} \frac{\partial^2 V}{\partial q^2} , \quad (15.7)$$

and requires them to be small. For the potential (15.4) at small U we find that $\epsilon \sim U^4$ and $\eta \sim U^2$ for all t . At large t there is further suppression of these parameters from the fact that $I(t)$ is exponentially small: this is obvious from the graphs in Figure 8.

For any U we can make t large enough that $1 \gg |\phi(t)|$. Using (13.11) we see that this is the case for

$$t \gg \frac{3}{2} \ln |U| ; \quad r^2 \varepsilon^{-4/3} \gg |U| . \quad (15.8)$$

Then

$$\frac{\gamma}{T_3} V(t) = \frac{U^2}{1 + e^{-\phi(t)}} \approx \frac{U^2}{2} + \frac{U^2}{4} \phi(t) \approx \frac{U^2}{2} - 3 \frac{U^4}{256} (4t - 1) e^{-4t/3} . \quad (15.9)$$

¹³ This is a standard notation in the cosmology literature; this ϵ should not be confused with the deformation parameter of the conifold.

Using this expression, we find ¹⁴

$$-V^{-1} \frac{\partial^2 V}{\partial r^2} \sim U^2 \varepsilon^{-4/3} (5t - 8) e^{-2t} . \quad (15.10)$$

Clearly, for any U this becomes very small at large t . To estimate the range of U for which the slow roll conditions are obeyed, we need to model a typical compactification. For this purpose, we introduce a cut-off at a large value of the radius, t_{UV} , where we find the scale of order $(\alpha')^{-1/2}$. Since $\frac{\varepsilon^{2/3}}{\alpha'}$ is the scale at the bottom of the inflationary throat, we have¹⁵

$$(\alpha')^{-1/2} \sim \frac{\varepsilon^{2/3}}{\alpha'} e^{t_{UV}/3} . \quad (15.11)$$

It is necessary that $\exp(t_{UV}/3)$ is a large factor: say, 4×10^3 as found in Appendix C of [17]. Let us assume that M_{Pl}^2 is comparable to the string scale $(\alpha')^{-1}$, up to a factor that is not very large (this is what happens for the numbers adopted in Appendix C of [17]). Then we have

$$|\eta| = -\frac{M_{Pl}^2}{T_3} V^{-1} \frac{\partial^2 V}{\partial r^2} \sim U^2 e^{2t_{UV}/3} e^{-2t} . \quad (15.12)$$

Requiring that this is much smaller than 1 for t around t_{UV} implies that

$$e^{4t_{UV}/3} \gg U^2 . \quad (15.13)$$

This is the same as the requirement that $\phi(t_{UV})$ is close to zero, (15.8); hence our treatment appears to be self-consistent. The slow roll condition $1 \gg |\eta|$, translated into (15.13), leaves a very large range of U available to modeling of inflation. The same is true for $1 \gg \epsilon$.

Hence, for a D3-brane moving on a resolved warped deformed conifold there is no difficulty in achieving very small values of ϵ and $|\eta|$ required for the slow-roll inflation. This suggests a D-brane inflation model similar to that of KKLM MT [17] (for earlier ideas in this direction, see [35]), but without the necessity of an anti-D3-brane at $t = 0$. However,

¹⁴ In the supergravity solution the asymptotic flatness of the potential is due to the fact that all baryonic branch backgrounds asymptote to the KT solution [4], where a D3-brane experiences no force. This mechanism for generating asymptotically flat potentials should apply to warped cones more general than the conifold. For example, for the warped cones over $Y^{p,q}$ found in [34] the D3-brane is BPS. Resolution of the naked singularity present at small radius may again lead to variation of the dilaton and generation of a potential for a D3-brane. But at large radius the solution has to asymptote to that of [34], so the force on a D3-brane will vanish asymptotically.

¹⁵ We will keep track of the exponential terms only, and ignore powers of t .

one also has to make sure that there are no additional, steeper corrections to the potential, that are introduced by the compactification effects.

Let us compare the scale of inflation in our construction with that in the KKLM model. In the KKLM model the asymptotic value of the potential is given by the tension of an anti-D3-brane placed at $t = 0$ in the KS solution, i.e.

$$\frac{2T_3}{H_{KS}(0)} = 2T_3 \frac{2^{8/3}}{\gamma I(0)} \approx 8T_3 \frac{2^{2/3}}{0.71805\gamma} . \quad (15.14)$$

In our construction, the asymptotic value of the potential is $T_3 U^2/(2\gamma)$, which is of the same order as (15.14) for U of order 1. Note that the large suppression relative to T_3 is due to the factor $\gamma^{-1} \sim e^{-4t_{uv}/3}$.

For $U \neq 0$ the D3-brane is eventually attracted to $t = 0$, and its tension there is

$$V(0) = \frac{T_3}{\gamma} \frac{U^2}{e^{-\phi_0} + 1} . \quad (15.15)$$

The plot of this quantity as a function of U is shown in Figure 7. It vanishes at $U = 0$, but for large $|U|$ it grows as $|U|^{5/4}$. Obviously, the addition of a D3 brane makes the scalar mode massive and lifts the baryonic branch, in agreement with the conclusions from section 7. The minimum of the potential is at the \mathbb{Z}_2 symmetric KS point, $U = 0$. Even though U is not a flat direction, let us imagine making $U \neq 0$ by hand, and adding a D3-brane at large t . The parameter U induces the FI term in the $U(1)$ gauge theory on the D3-brane, whose gauge coupling is g_{YM} . Hence, roughly, the flat potential

$$\frac{T_3 U^2}{2\gamma} = \frac{1}{2^{2/3} \pi^3 (g_s M)^2} \frac{U^2}{g_s} \frac{\varepsilon^{8/3}}{(\alpha')^4} \quad (15.16)$$

at infinity originates from the $\frac{1}{2} g_{YM}^2 \xi^2$ D-term in this $U(1)$ gauge theory. Indeed, identifying ξ with $\langle \mathcal{U} \rangle$ from (13.8), we find

$$\frac{1}{2} g_{YM}^2 \xi^2 \sim g_s \xi^2 \sim (g_s M)^2 \frac{U^2}{g_s} \frac{\varepsilon^{8/3}}{(\alpha')^4} . \quad (15.17)$$

Up to a factor involving powers of $g_s M$, this agrees with the probe D3-brane result (15.16). This extra factor may appear in extrapolating from small to large values of $g_s M$ (see section 11.3 for a discussion of similar effects).

In the above discussion, the deformation to nonzero U was put in by hand. In other words, there is a potential for U which pushes it to the origin where supersymmetry is

restored. However, when the throat is embedded into a warped compactification, as in [25], the $U(1)_{baryon}$ becomes gauged, and we may add a Fayet-Iliopoulos term ξ for it [18]. (There has been a lot of discussion about Fayet-Iliopoulos terms in string theory and supergravity. Two typical examples are [36,37].) The Fayet-Iliopoulos term should force the throat background to a non-zero value of $U \sim \frac{\xi(\alpha')^2}{M\epsilon^{4/3}}$.

Therefore, to construct a real model of D-brane inflation we need to consider a flux compactification with a non-vanishing ξ , whose throat region is described by a resolved warped deformed conifold, and add a D3-brane. This makes our construction similar to the D-term inflation of [19,20]. That model assumes a $U(1)$ gauge theory coupled to chiral superfields X , ϕ_+ , and ϕ_- of charges 0, 1 and -1 , respectively. Then one adds the superpotential $W = \lambda X \phi_+ \phi_-$ and turns on the Fayet-Iliopoulos term ξ . For sufficiently large $|X|$, and with $\phi_{\pm} = 0$, the potential is found to be [19,20]

$$V_{eff} = \frac{1}{2} g^2 \xi^2 \left(1 + \frac{g^2}{16\pi^2} \ln \frac{\lambda^2 |X|^2}{\Lambda^2} + O(g^4) \right). \quad (15.18)$$

This model is similar to our construction for the $SU(1 + M(k + 1)) \times SU(1 + Mk)$ gauge theory with the gauged $U(1)_{baryon}$. In the simplest case of $k = 0$ we find $SU(M + 1) \times U(1)_{baryon}$ gauge theory already discussed in sections 5 and 10. In our construction the meson fields $\mathcal{M}_{\alpha\dot{\alpha}}$ are the analogues of the neutral field X , and the charged fields $A_{\alpha a}$, $B_{\dot{\alpha}}^b$ are the analogues of ϕ_{\pm} .

We can clearly see the asymptotically constant term in the potential in our probe brane calculation, but we do not observe the logarithmic one-loop correction. However, in a flux compactification, various additional corrections to the potential should appear. In fact, as pointed out in [17], the effects of compactification could make the potential significantly steeper than what the pure throat limit (15.9) indicates. Investigation of the effects of compactification is beyond the scope of this paper, but we hope to address them in the future.

To summarize, our proposal for stringy D-term inflation proceeds as follows. We consider a warped compactification with fluxes, which has a warped deformed conifold region. Then we turn on the Fayet-Iliopoulos term for the gauged $U(1)_{baryon}$ symmetry, which forces a breaking of the \mathbb{Z}_2 symmetry. As explained in [8,18] the throat limit of such backgrounds is provided by the resolved warped deformed conifolds, that were later constructed in [10]. Then we add a D3-brane that breaks supersymmetry, and show that its potential as a function of the radius varies slowly, at least in the gauge theory limit. Near

$t = 0$ the potential gets steeper, and the D3-brane accelerates. After reaching $t = 0$ the D3-brane will undergo oscillations and internal vibrations which could reheat the Universe. After the D3-brane stabilizes at $t = 0$ in a background with non-vanishing U , it makes a positive contribution to the vacuum energy. This positive contribution could be used to cancel the negative contribution to cosmological constant that arises through a non-perturbative mechanism of the type suggested in [38]. As a result, the net cosmological constant can be made small and positive, although this may require fine-tuning as usual. Therefore, our approach appears to avoid the necessity of an anti-D3-brane [38,17] or a D7-brane [21] that played important roles in earlier constructions; instead, we use a D3-brane on a *resolved* warped deformed conifold. We leave a detailed investigation of this model for the future.

16. Discussion

We have found that the cascading $SU(N_1) \times SU(N_2)$ gauge theory has many branches of the moduli space. It would be interesting to extend our systematic study of the moduli space to more complicated cascading theories, for example to theories on D-branes near the tip of the cone over $Y^{p,q}$. Some results on the asymptotic structure of the cascade are already available [34,39], while in the infrared different possibilities have been suggested: dynamical SUSY breaking or runaway behavior where the supersymmetry is restored [40–42]. In fact, it is possible that some of the branches of the moduli space lead to dynamical SUSY breaking while others to runaway behavior. Improved understanding of these issues should facilitate work on finding IR completions of the cascading solution found in [34].

Another interesting direction raised by our work is to embed the D3-branes on a resolved warped deformed conifold into a string compactification with a Fayet-Iliopoulos term. Such a model could be a useful variation on the KKLM model. These inflationary models have natural generalizations to D3-branes rolling on other cascading geometries (for example, those found in [34]) embedded into flux compactifications.

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Appendix A. Review of the Papadopoulos-Tseytlin Ansatz

The PT ansatz describes a warped product of the 4-dimensional flat space $\mathbb{R}^{3,1}$ and a non-compact six-dimensional manifold M^6 , which is roughly speaking the deformed conifold with some internal warp factors (12.2).

The field strengths are [9] (the forms ϵ_i and e_i are defined below)

$$\begin{aligned}
H_3 &= h_2(t) \tilde{\epsilon}_3 \wedge (\epsilon_1 \wedge e_1 + \epsilon_2 \wedge e_2) + dt \wedge [h'_1(t)(\epsilon_1 \wedge \epsilon_2 + e_1 \wedge e_2) \\
&\quad + \chi'(t)(-\epsilon_1 \wedge \epsilon_2 + e_1 \wedge e_2) + h'_2(t)(\epsilon_1 \wedge e_2 - \epsilon_2 \wedge e_1)] , \\
F_3 &= P \tilde{\epsilon}_3 \wedge [\epsilon_1 \wedge \epsilon_2 + e_1 \wedge e_2 - b(t)(\epsilon_1 \wedge e_2 - \epsilon_2 \wedge e_1)] \\
&\quad + dt \wedge [b'(t)(\epsilon_1 \wedge e_1 + \epsilon_2 \wedge e_2)] , \\
g_s F_5 &= \mathcal{F}_5 + *_{10} \mathcal{F}_5 , \quad \mathcal{F}_5 = K(t) e_1 \wedge e_2 \wedge \epsilon_1 \wedge \epsilon_2 \wedge \epsilon_3 .
\end{aligned} \tag{A.1}$$

The six-dimensional manifold M_6 has the topology of $\mathbb{R}^1 \times SU(2) \times SU(2)/U(1) = \mathbb{R}^1 \times \mathbb{S}^2 \times \mathbb{S}^3$ and the variable t parameterizes the \mathbb{R}^1 . The forms $\{e_1, e_2\}$ correspond to \mathbb{S}^2 , while the forms $\{\epsilon_1, \epsilon_2, \epsilon_3\}$ are the left-invariant forms on \mathbb{S}^3 as we will see below. The space at constant t approaches $T^{1,1}$ in the UV region $t \rightarrow \infty$. In fact, the UV asymptotic metric is $AdS_5 \times T^{1,1}$ modulo slowly-varying logarithms [4] which are present due to the logarithmic RG flow in the dual gauge theory [3].

The description that makes the $SU(2) \times SU(2)$ symmetry explicit defines M^6 via an algebraic equation

$$\begin{aligned}
det W &= -\frac{\varepsilon^2}{2} , & W &= \rho(t) U_1 Z U_2^+ , \\
U_i &= \begin{pmatrix} a_i & b_i \\ -b_i^* & a_i^* \end{pmatrix} \in SU(2) , & Z &= \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix} , \\
a_i &= \cos(\theta_i/2) e^{i(\psi_i + \phi_i)/2} , & b_i &= \cos(\theta_i/2) e^{i(\psi_i - \phi_i)/2} , \\
\rho(t) &= \frac{\varepsilon e^{-t/2}}{\sqrt{2}} \sqrt{1 + e^{2t}} , & \alpha &= \frac{e^t}{\sqrt{1 + e^{2t}}} , & \beta &= \frac{1}{\sqrt{1 + e^{2t}}} .
\end{aligned} \tag{A.2}$$

Then one gauges the $U(1)$ symmetry that acts by $\psi_i \rightarrow \psi_i + (-1)^i C$ and introduces the invariant combination $\psi = \psi_1 + \psi_2$ (ψ could be also understood as ψ_2 when $\psi_1 = 0$). Now, we introduce the invariant forms ϵ_i (σ_i is the Pauli matrix)

$$\begin{aligned} 2\epsilon_i &= \text{Tr}(U_2^+ dU_2 \sigma_i) , \\ \epsilon_1 &\equiv \sin \psi \sin \theta_2 d\phi_2 + \cos \psi d\theta_2 , \\ \epsilon_2 &\equiv \cos \psi \sin \theta_2 d\phi_2 - \sin \psi d\theta , \\ \epsilon_3 &\equiv d\psi + \cos \theta_2 d\phi_2 , \end{aligned} \tag{A.3}$$

and $2\hat{\epsilon}_i = \text{Tr}(U_1^+ dU_1 \sigma_i)$. The combination

$$\tilde{\epsilon}_3 = \epsilon_3 + \hat{\epsilon}_3 = d\psi + \cos(\theta_1) d\phi_1 + \cos(\theta_2) d\phi_2 \tag{A.4}$$

is therefore also invariant under $SU(2) \times SU(2)$.

In the original form [9], the PT ansatz uses $SU(2)_L$ non-invariant forms

$$e_1 \equiv d\theta_1 , \quad e_2 \equiv -\sin \theta_1 d\phi_1 , \tag{A.5}$$

rather than the invariant \hat{e}_1, \hat{e}_2 . But e_1, e_2 appear only in combinations that could be represented via \hat{e}_1, \hat{e}_2 . For this sake we introduce $SU(2)_R$ non-invariant forms $\hat{e}_1 \equiv d\theta_2$, $\hat{e}_2 \equiv -\sin \theta_2 d\phi_2$ and then express $SU(2)_R$ explicitly invariant LHS via explicitly $SU(2)_L$ invariant RHS

$$\begin{aligned} e_1^2 + e_2^2 &= \hat{e}_1^2 + \hat{e}_2^2 , \\ e_1 \epsilon_1 + e_2 \epsilon_2 &= \hat{e}_1 \hat{e}_1 + \hat{e}_2 \hat{e}_2 , \\ e_1 \wedge \epsilon_1 + e_2 \wedge \epsilon_2 &= \hat{e}_1 \wedge \hat{e}_1 + \hat{e}_2 \wedge \hat{e}_2 , \\ e_1 \wedge \epsilon_2 - e_2 \wedge \epsilon_1 &= -\hat{e}_1 \wedge \hat{e}_2 + \hat{e}_2 \wedge \hat{e}_1 , \\ e_1 \wedge e_2 &= -\hat{e}_1 \wedge \hat{e}_2 . \end{aligned} \tag{A.6}$$

The PT ansatz is $SU(2) \times SU(2)$ invariant but in general breaks the \mathbb{Z}_2 symmetry that interchanges e_1, e_2 with ϵ_1, ϵ_2 . This \mathbb{Z}_2 symmetry is restored for the warped deformed conifold solution of [5] by virtue of the identity $e^g + a^2 e^{-g} = e^{-g}$ as seen from (12.2).

Appendix B. First-order equations

The functions $a, g, x, v, A, h_1, h_2, \chi, K$ depend on the radial variable t only. The crucial result of [10] is that supersymmetry of the background requires $a(t)$ and $v(t)$ to satisfy the

coupled first-order equations

$$\begin{aligned}
a' &= - \frac{\sqrt{-1 - a^2 - 2a \cosh t} (1 + a \cosh t)}{v \sinh t} - \frac{a \sinh t (t + a \sinh t)}{t \cosh t - \sinh t}, \\
v' &= \frac{-3a \sinh t}{\sqrt{-1 - a^2 - 2a \cosh t}} + \\
&\quad + v \left[-a^2 \cosh^3 t + 2at \coth t + a \cosh^2 t (2 - 4t \coth t) + \cosh t (1 + 2a^2 \right. \\
&\quad \left. - (2 + a^2) t \coth t) + \frac{t}{\sinh t} \right] / \left[(1 + a^2 + 2a \cosh t) (t \cosh t - \sinh t) \right].
\end{aligned} \tag{B.1}$$

The solution of these equations determines other unknown functions through (12.3) and

$$\begin{aligned}
e^{2g} &= -1 - a^2 + 2aC, \\
e^{2x} &= \left(\frac{g_s M \alpha'}{2} \right)^2 \frac{(bC - 1)^2}{4(aC - 1)^2} e^{2g+2\phi} (1 - e^{2\phi}), \\
b &= -\frac{t}{\sinh(t)}, \\
h_1 &= -Ch_2, \\
h_2 &= \left(\frac{g_s M \alpha'}{2} \right) \frac{e^{2\phi}(bC - 1)}{2S}, \\
\chi' &= \left(\frac{g_s M \alpha'}{2} \right) a(b - C)(aC - 1) e^{2(\phi-g)}, \\
K &= - \left(\frac{g_s M \alpha'}{2} \right) (h_1 + bh_2) = \left(\frac{g_s M \alpha'}{2} \right)^2 e^{2\phi} \frac{(bC - 1)(C - b)}{2S}, \\
P &= - \left(\frac{M \alpha'}{4} \right),
\end{aligned} \tag{B.2}$$

where $C = -\cosh(t)$, $S = -\sinh(t)$.

We have fixed normalizations from the condition that there are M units of flux of the RR 3-form field strength F_3 through the \mathbb{S}^3

$$\frac{1}{4\pi^2 \alpha'} \int_{\mathbb{S}^3} F_3 = M. \tag{B.3}$$

The integer M is dual to the difference between the numbers of colors of the two gauge groups.

Since (B.1) is a system of two coupled first-order equations, one might expect a two-parameter family of solutions, but in fact all solutions regular at $t = 0$ are parameterized

by just one real parameter y . The small t expansion found in [10] is

$$\begin{aligned} a &= -1 + \left(\frac{1}{2} + \frac{y}{3}\right)t^2 + \dots, \\ v &= t + \left(-\frac{13}{96} + \frac{17y^2}{216}\right)t^3 + \dots \end{aligned} \tag{B.4}$$

The parameter ξ_{BGMPZ} defined as $\xi_{BGMPZ} = 1/2 + y/3$ varies from $1/6$ to $5/6$ when y varies from -1 to 1 along the baryonic branch. Any value ξ_{BGMPZ} is related to $1 - \xi_{BGMPZ}$ by the \mathbb{Z}_2 symmetry $y \rightarrow -y$ which changes the sign of U but leaves $v(t)$ and the combination $a(t)e^{-g(t)}$ invariant. The \mathbb{Z}_2 symmetric value $y = 0$ obviously corresponds to the KS solution dual to the locus on the baryonic branch where $|\mathcal{A}| = |\mathcal{B}| = \Lambda_1^{2M}$ and $e^{2g} + a^2 = 1$.

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